LoP- The Logic of Propositions

Reasoning about what is True and what is False

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- Introduction
- Domain
- Language
- Interpretation function
- Entailment
- Model and theory
- Entailment properties
- Modeling mistakes
- Reasoning problems

Knowledge Bases (reprise)

The original use of the term knowledge base was to describe one of the two sub-systems of an expert system (a knowledge-based system).

A knowledge-based system [*] consists of

- a knowledge-base representing *facts* about the world and (LODE)
- ways of reasoning about those *facts* to deduce new facts or highlight inconsistencies. (LOP)

[*] Hayes-Roth, Frederick; Donald Waterman; Douglas Lenat (1983). Building Expert Systems. Addison-Wesley.

Facts, Assertions, Definitions, … (reprise)

- We **depict** the world as a set of **facts** (Set, domain, model, *data and knowledge level depictions of the world*)
- We **structure** facts in terms of entities, types, properties (*data or knowledge level depictions of the world*)
- We **describe** facts (involving entities, types, properties) in the world using **assertions** (LoE, language, theory, *data level atomic assertions, descriptions of the world*)
- We **define** and **inter-relate** the concepts (i.e., the meaning of the words) we use in assertions using **definitions**. This allows us to describe facts at different levels of abstraction (LoD, definitions, *knowledge level complex formulas, descriptions of the world*)

... (Populated) Descriptions, Propositions (reprise)

- We **describe** the diversity/ variability of concepts using **descriptions** (LoD, descriptions, *knowledge level complex formulas*)
- We **describe** the diversity/ variability of entities populating concepts using **grounded descriptions** (LoDE, *data level complex formulas*)
- We **reason about** grounded descriptions using **propositions** (LoP, *truth level complex formulas*)

Propositions

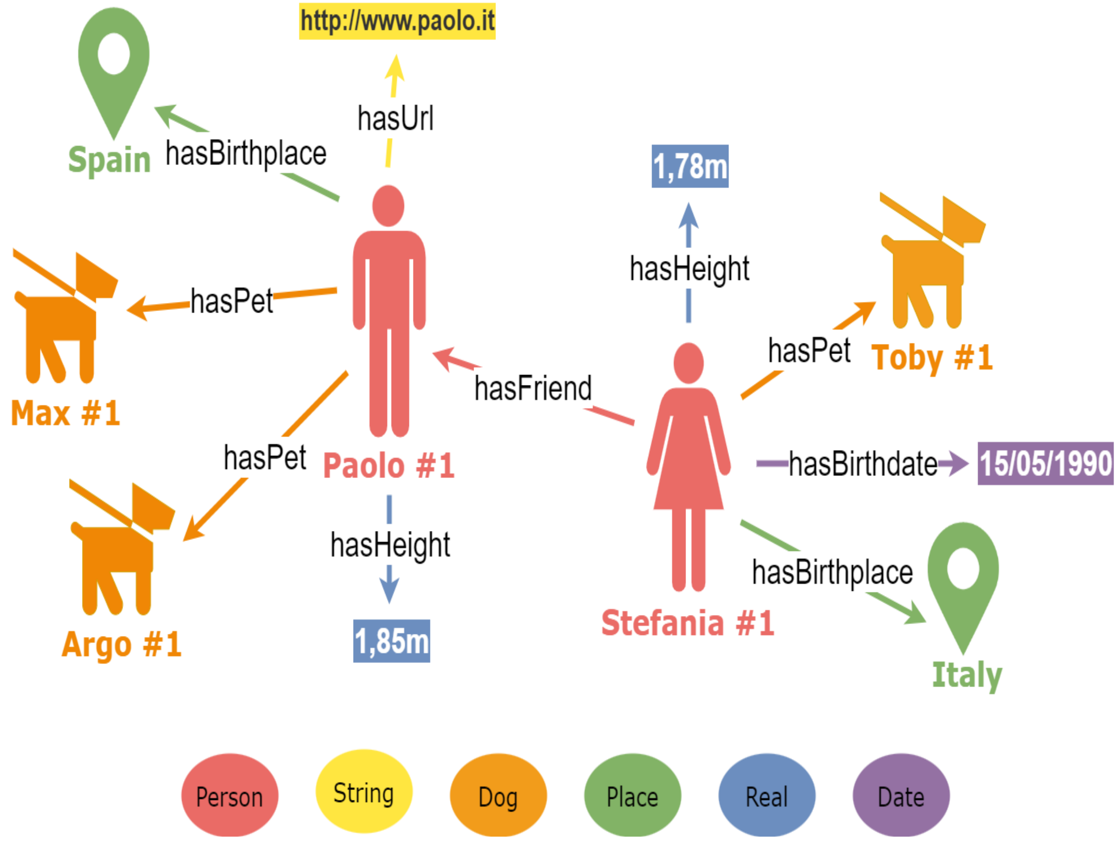
Notion (Google/ Oxford Languages). A **proposition** is an assertion that expresses a judgement or opinion.

Notion (Aristotle). A **proposition** is a sentence which affirms or denies a predicate of a subject.

Notion (LoP). A **proposition** is a formula which can be either true or false; it must be one or the other (*Law of excluded middle*), and it cannot be both (*Law of noncontradiction*).

Observation. Representing Truth/Falsity is the key for implementing reasoning.

An example of EG



Which of the following assertions are intuitively true?

- *HasFriend(Paolo#1,Stefania#1)*
- *Hasheight(Stefania#1, 2m)*
- *HasPet (Stefania#1, Fido#1)*
- *Not HasHeight(Stefania#1, 2m)*
- *HasFriend(Paolo#1,Stefania#1) and HasHeight(Stefania#1, 2m)*
- *HasFriend(Paolo#1,Stefania#1) or HasHeight(Stefania#1, 2m)*
- *HF*
- *HF and HH*
- *HF or HH*
- *....*

Which Interpretation function?

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LoP (= LoDE) – Domain

Definition (LoE/ LoD/ LoDE Domain)

$$D = < E, \{C\}, \{R\} >$$

with:

$$E = \{e\} \cup \{v\}$$

$$\{C\} = ET \cup DT \cup DET$$

$$\{R\} = \{OR\} \cup \{DR\}$$

where:

- E is a set of **entities** and **values**,
- $ET = \{E_T\}$, $E_T = \{e\}$ and $DT = \{D_T\}$, $D_T = \{v\}$, $DET = \{DE_T\}$, are **sets of etypes, dtypes**, and **defined etypes**, respectively
- OR, DR are **(binary) object** and **data relations**.

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Language

Definition (The language of LoP)

$$\text{L} = \text{L}a \cup \text{L}c$$

Definition (The language of atomic formulas La)

$$\text{L}a =< \text{A}a, \emptyset >$$

Definition (Alphabet Aa)

$$\text{A}a =< \{P\} >$$

Where $P \in \{P\}$ is a **proposition**.

Observation: There are no formation rules for atomic formulas. Propositions are judgements about facts, without references to their internal structure, i.e., the entities and relations that compose them. ***The only interest** is to reason about truth!*

Language (cont) - Lc

<cwff> ::= <proposition> |
¬ <cwff> |
<cwff> ∧ <cwff> |
<cwff> ∨ <cwff> |
<cwff> ⊃ <cwff> |
<cwff> ≡ <cwff>

<proposition> ::= P1 . . . Pn $\in \{P\}$

Where do P1, ..., Pn come from?

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LOP Interpretation function

Definition (LOP Interpretation function). Let D = < E, {C}, {R} > be a LoDE domain of interpretation. Let $L_{LODE} = L\mathcal{a}_{LODE} \cup L\mathcal{c}_{LODE}$ be a LoDE language for D. Let I_{LODE} be a LoDE interpretation function, with $I_{LODE}: L\mathcal{a}_{LODE} \rightarrow D$.

Let $L_{LOP} = L\mathcal{a}_{LOP} \cup L\mathcal{c}_{LOP}$. Let $P \in \{P\}$ be a LOP proposition, with $\{P\} = L\mathcal{a}_{LOP}$. Let I_{LOP} be a LoP interpretation function, with $I_{LOP}: \{P\} \rightarrow \{T,F\}$.

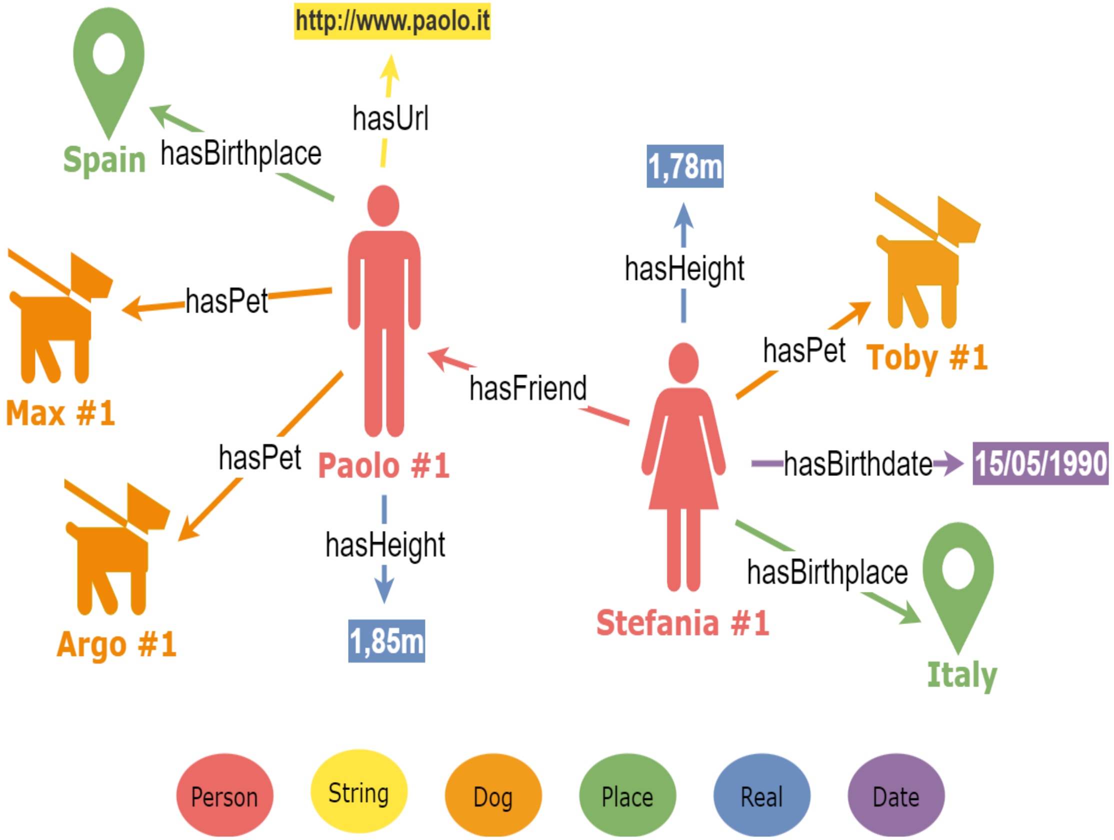
Let *Translate* a bijective (injective and surjective) function such that

$$\text{for all } A \in L\mathcal{a}_{LODE},\ Translate(A) = P_A \text{ with } P_A \in \{P\}$$

Then we have the following

- $\mathbf{I_{LOP}(P) = T}$ se e solo se $I_{LODE}(Translate^{-1}(P)) \in M \quad (\Leftarrow\Rightarrow M \models_{LODE} Translate^{-1}(P))$
- $\mathbf{I_{LOP}(P) = F}$ se e solo se $I_{LODE}(Translate^{-1}(P))$ *NOT* $\in M$

Example – which truth values of which propositions



- *HasFriend(Paolo#1,Stefania#1)*
- *Hasheight(Stefania#1, 2m)*
- *HasPet (Stefania#1, Fido#1)*
- *Not HasHeight(Stefania#1, 2m)*
- *HasFriend(Paolo#1,Stefania#1) and HasHeight(Stefania#1, 2m)*
- *HasFriend(Paolo#1,Stefania#1) or HasHeight(Stefania#1, 2m)*

…

Intepretation as set of propositions

Observation 2. An interpretation I can be represented set theoretically as the set of true propositions it defines as true).

Example:

	p	q	r	Set Theoretic Representation
I_1	True	True	True	$\{p, q, r\}$
I_2	True	True	False	$\{p, q\}$
I_3	True	False	True	$\{p, r\}$
I_4	True	False	False	$\{p\}$
I_5	False	True	True	$\{q, r\}$
I_6	False	True	False	$\{q\}$
I_7	False	False	True	$\{r\}$
I_8	False	False	False	$\{\}$

Observation 3. A propositional interpretation can be thought as a subset S of $\{P\}$ and I is the characteristic function of S, i.e.

$$A \in S \text{ if and only if } I(A) = True.$$

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LOP Model and theory

Definition (LOP Theory). A LOP theory is a set of formulas

$$w \in T \subseteq L$$

(similarly to LODE theories).

Definition (LOP Model). A LOP model is a set of propositions

$$\{P_T\}, \subseteq \{P\},$$

with $\{P\}$ = La_{LOP} (differently from LODE theories)

LOP Entailment $\vDash$

$M \vDash P$,	if $I(P)$ = T,	with	$P \in \{P\}$
$M \vDash \neg P$,	if	not	$M \vDash P$
$M \vDash P_1 \wedge P_2$,	if $M \vDash P_1$	and	$M \vDash P_2$
$M \vDash P_1 \vee P_2$,	if $M \vDash P_1$	or	$M \vDash P_2$
$M \vDash P_1 \supset P_2$,	if when $M \vDash P_1$,	then	$M \vDash P_2$
$M \vDash P_1 \equiv P_2$,	if $M \vDash P_1$	if and only if	$M \vDash P_2$

Observation: How Connectives Operate

Truth values are both in input and output to connectives

Negation	
¬ True	False
¬ False	True

Conjunction	
True ∧ True	True
True ∧ False	False
False ∧ True	False
False ∧ False	False

Consequence	
True ⊃ True	True
True ⊃ False	False
False ⊃ True	True
False ⊃ False	True

Disjunction	
True V True	True
True V False	True
False V True	True
False V False	False

Equivalence	
True ≡ True	True
True ≡ False	False
False ≡ True	False
False ≡ False	True

LOP Entailment – Negation

Observation 1. The key intuition underlying LOP (and therefore how we model reasoning) is that reasoning is completely independent of how we ascertain the truth of atomic formulas (assertions in LODE, propositions in LOP).

Observation 2. The real world (that is, analogic representations) only tells us the truth of assertions. Once we have that, reasoning is only linguistic and independent of what is the case in the world.

Observation 3. The key difference, with respect to LODE is that in LOP it is possible to assert the falsity of a proposition (in LODE one can only assert true facts).

Observation 4. Any fact that in LODE is not asserted as being true may be taken to be true or false in LOP. This capturing the fact that whether this fact is true/ false, is unknown.

LOP Entailment - negation

Observation 5. The meaning of negation is given by the law of the excluded middle and the law of contradiction

The law of the excluded middle

$$P \vee \neg P$$

- True in all models.
- All formulas of the above form, independently of the shape of P, are called ***tautologies***.
- Sometimes they are written as ***T*** (for truth, as represented in the language)
- The interpretation of ***T*** is T

Prove them!

The law of noncontradiction

$$P \wedge \neg P$$

- Never true, in no model.
- All formulas of the above form, independently of the shape of P, are called ***contradictions***.
- Sometimes they are written as ⊥ (for falsity, as represented in the language). Not to be confused with ⊥ (bottom) in LOE!
- The interpretation of ⊥ is F

LOP Entailment - conjunction/disjunction

Same proposition
- $A \wedge A \equiv A$
- $A \vee A = A$

Commutativity
- $A \wedge B \equiv B \wedge A$
- $A \vee B \equiv B \vee A$

De Morgan laws
- $\neg (A \vee B) \equiv \neg A \wedge \neg B$
- $\neg (A \wedge B) \equiv \neg A \vee \neg B$

Associativity
- $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$
- $(A \vee B) \vee C \equiv A \vee (B \vee C)$

Distributivity
- $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$
- $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

Prove them!

Which intuition in LODE semantics?

LOP Entailment - implication/equivalence

Implication and disjunction

- $(A \supset B) \equiv (\neg A \vee B)$

Implication and negation

- $A \supset B \equiv \neg B \supset \neg A$

Implication and contradiction

- $\bot \supset A$, for any A

Implication and equivalence

- $(A \equiv B) \equiv ((A \supset B) \wedge (B \supset A))$

Equivalence and exclusive disjunction (exor)

- $(A \equiv B) \equiv \neg(A + B)$

Exclusive and inclusive disjunction

- $(A + B) \equiv (\neg A \wedge B) \vee (A \wedge \neg B)$

Prove them!

Which intuition in LODE semantics?

LOP Entailment - implication/conj/disj

Implication and conjunction (1)

- $(A \wedge B) \supset C \equiv (A \supset C) \vee (B \supset C)$

Implication and conjunction (2)

- $(A \wedge B) \supset C \equiv A \supset (B \supset C)$

Implication and conjunction (3)

- $(A \wedge B) \supset C \equiv A \supset (\neg B \vee C)$

Implication and conjunction (4)

- $A \supset (B \wedge C) \equiv (A \supset B) \wedge (A \supset C)$

Implication and disjunction (1)

- $(A \supset (B \vee C) \equiv (A \supset B) \vee (A \supset C)$

Implication and disjunction (2)

- $(A \vee B) \supset C \equiv (A \supset C) \wedge (B \supset C)$

Prove them!

Which intuition in LODE semantics?

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LOP Model and interpretation

Observation 1: A LOP model is the set of propositions P ∈ $\{P\}$ such that I_{LOP} (P) = T, namely the set of propositions corresponding (via *Translate*) to assertions which are true in the LODE model.

Observation 2: In LOP, the notion of model and that of interpretation are collapsed and it is said that "… a model is an interpretation which makes true all the formulas in a theory T".

Observation 3: A LOP model assigns to any LOP atomic formula either T or F. For any LODE model the corresponding LOP model is obtained by adding negative propositions for those assertions do not hold in the LODE Model.

LOP Entailment – From models to interpretations

$I \vDash P$,	if $I(P)$ = T,	with	$P \in \{P\}$
$I \vDash \neg P$,	if	not	$I \vDash P$
$I \vDash P_1 \wedge P_2$,	if $I \vDash P_1$	and	$I \vDash P_2$
$I \vDash P_1 \vee P_2$,	if $I \vDash P_1$	or	$I \vDash P_2$
$I \vDash P_1 \supset P_2$,	if when $I \vDash P_1$,	then	$I \vDash P_2$
$I \vDash P_1 \equiv P_2$,	$I \vDash P_1$	if and only if	$I \vDash P_2$

Interpretation equivalence wrt. a formula

Observation 4. If for all and only the atomic propositions P occurring in a formula A we have:

$$I(P) = I'(P),$$

then

$$I \vDash A \text{ iff } I' \vDash A.$$

That is:

- The truth value of atomic propositions which occur in A fully determines the truth value of A
- The truth value of the atomic propositions which do not occur in A play no role in the computation of the truth value of A;

Model and theory, observations

Observation 5 (Maximum number of models for a LOP language). If $|\{P\}|$ is the cardinality of $\{P\}$, then there are $2^{|\{P\}|}$ different models, corresponding to all the different subsets of $\{P\}$.

Observation 6 (Number of theories for a model). A LOP model can be described by multiple theories, all assigning the same truth values to propositions.

Observation 7 (Number of models of a theory). A theory T has usually multiple models. T can have any number of models between 0 (when it contains a contradiction) and $2^{|\{P\}|}$ when all its formulas are tautologies

Observation 8 (Maximal theories). Some theories have only one model. These theories and EG's are called *maximal,* or *complete*. For instance, T1= $\{P_1 \wedge \neg P_2\}$, and T2= $\{P_1 \wedge P_2\}$, with $\{P\} = \{P_1, P_2\}$ are two maximal theories. A model has multiple maximal theories (as from observation 7)

Model and theory, observations

Observation 9 (Maximal LOP theories and completeness of LODE EG's). A maximal LOP theory can be generated by translating a LODE EG (see the definition of LOP interpretation function). These theories correspond to EG's which are complete, namely which assert that all propositions (and therefore all assertions allowed by the language of the EG) are true.

Observation 10 (Maximal LOP theories and partiality of LODE EG's). A maximal LOP theory which asserts the falsity of a proposition can only be generated by stating the falsity of an assertion not stated in the LODE EG.

Observation 11 (Maximal LOP theories and partiality of LODE EG's). A maximal LOP theory can also be generated by asserting the truth of the missing assertion. Any missing LODE assertion in fact doubles the number of interpretations (that is, potential LOP models).

Observation 12 (Partial LOP theories and partiality of LODE EG's). A partial LOP theory can be defined by generating one true proposition for any LODE assertion. Any missing assertion will double the number of interpretations of the resulting LOP partial theory

Model and theory, observations

Observation 14 (Partiality of LOP theories). The more partial a LOP theory T is, in terms of truth values assigned to propositions, the more models. For instance, assume $\{P\}$ = $\{P_1, P_2\}$.

- T = $\{P1 \vee \neg P1, P2 \vee \neg P2\}$, has four models
- T = $\{P1 \vee \neg P2\}$, has three models
- T = $\{P1\}$ has two models
- T = $\{P1 \wedge P2\}$ has one model
- T = $\{P1 \wedge \neg P2\}$ has no models

Observation 13 (Partiality of LODE EG's). An increase in the partiality of a LODE EG causes an exponential increase of the number of LOP models (see above). At the same time it allows for more compact LODE EG's. In fact you can use a set of propositions $\{P\}$ which describe what is relevant. Thus for instance you can have $\{P\}$ = {Tall} instead of $\{P\}$ = {Tall, Short} with Tall ≡ ¬Short. Short is added to the EG only if there is a need to describe entities which are short, as well as entities which are Tall

Minimal models, observations

Observation 16. Given a theory T, there is no *minimal* Model M which is the intersection of all models of T, the main cause being disjunction. For instance, assume $\{P\} = \{P_1, P_2\}$. T= $\{P_1 \vee P_2\}$ has four models and no minimal model.

Observation 17: LODE theories have minimal models (no disjunctions, single premises)

Entailment, Truth and Satisfiability

The following statements are **equivalent enunciations** of the statement $I \models A$:

- the interpretation function (model) I ***entails*** the formula A;
- the formula A *is* ***true*** in the interpretation function (model) I;
- the formula A is ***satisfied*** by the interpretation function (model) I.

Example: Let P and Q be two propositions: $\{P\}$ = {A, B}. I(A) = $True$ and I(B) = $False$ can be also expressed with I = {A}.

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Entailment properties (NEW!)

Deduction theorem (Logical consequence, validity):

$$\Gamma, \phi \vDash \psi \text{ if and only if } \Gamma \vDash \phi \supset \psi$$

Observation 1: The deduction theorem explains (left to right) the meaning of implication. Implication is how we express logical consequence in language.

Observation 2: It also says (right to left) that from absurdity (i.e, $P \wedge \neg P$),we can derive everything, any formula (and assertion) A.

Entailment properties (NEW!)

Refutation principle (Logical consequence, unsatisfiability):

$\Gamma \vDash \phi$ if and only if $\Gamma \cup \{\neg \phi\}$ is unsatisfiable

Observation 1: The refutation principle explains the meaning of negation. It captures the fact that absurdity (i.e, $P \wedge \neg P$) cannot be satisfied by any model depicting facts in the real world.

Observation 2: Algorithmitically, it suggests how to reason backwards from goals.

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Modeling mistakes – And (1)

We express conjunction with many words other than "and", including "but," "moreover," "however, "although", and "even though".

For example: "I enjoyed the holiday, even though it rained a lot" can be translated into the facts "I enjoyed the holiday" and "It rained a lot".

Sometimes "and" joins adjectives.

For example: "The leech was long and wet and slimy." This can be paraphrased as "The leech was long, and the leech was wet, and the leech was slimy.

Modeling mistakes – And (2)

Sometimes "and" does not join whole propositions into a compound proposition. Sometimes it simply joins nouns. This cannot be paraphrased. In these cases, the "and" is expressed inside the propositional variable, and not as logical connective.

For example: "Bert and Ernie are brothers". This cannot be paraphrased. "Bert is a brother and Ernie is a brother", for that does not assert that they are brothers to each other.

Modeling mistakes – Inclusive vs. Exclusive disjunction

The natural, but longwinded, way to express exclusive disjunction is $(\neg p \vee q) \wedge (p \vee \neg q)$.

The way to say they have different truth values is to deny their equivalence: $\neg (p \equiv q)$.

For example: When a menu says "cream or sugar", it uses an inclusive "or", because you may take one, the other, or both. But when it says "coffee or tea", it uses an exclusive "or", because you are not invited to take both.

Modeling mistakes – Implication

$p \supset q$ translates a wide variety of English expressions, for example, "if p, then q", "if p, q", "p implies q", "p entails q", "p therefore q", "p hence q", "q if p", "q provided p", "q follows from p", "p is the sufficient condition of q", and "q is the necessary condition of p". The least intuitive is " p only if q" (to be understood from $\neg q \supset \neg p$).

For example the following all translate to $p \supset q$:

- If Mario goes to the party, (then) I'll go too.
- I'll go to the party if/provided that Mario comes too.
- I'll go to the party only if Mario goes.
- Mario going to the party is the sufficient condition of me going to the party.
- Me going to the party is necessary condition of Mario going to the party.
- The decrease in white blood cells implies the antibiotic is working.

Modeling mistakes – Even If

"p even if q" means "p whether or not q" or "p regardless of q". Therefore one perfectly acceptable translation of it is simply "p". If you want to spell out the claim of "regardlessness", then you could write "$p \wedge (q \vee \neg q)$".

For example:

- I'll go to the party even if Mario doesn't go.
- I'll go to the party whether or not Mario goes.
- I'll go to the party regardless of whether Mario comes or not

Modeling mistakes – Unless

Sometimes "unless" should be translated as inclusive disjunction, and sometimes as exclusive disjunction.

For example (inclusive disjunction): "I’ll go to the party unless I get another offer" means that I’ll go if nothing else comes along, namely an exclusive disjunction. In many contexts it also means that I might go anyway; the second offer might be worse. So I’ll go or I’ll get another offer or both. Example: “I’ll go only to the party unless I get another offer”

For example (exclusive disjunction): Consider by contrast, "I’ll go to the party unless Rufus is there". In many contexts this means that if I learn Rufus is going, then I’ll change my mind and not go. So either I’ll go or Rufus will go but not both.

Modeling mistakes – Necessary and Sufficient Condition

We say that p is a sufficient condition of q when p's truth guarantees q's truth. By contrast, q is a necessary condition of p when q's falsehood guarantees p's falsehood.

In the ordinary material implication, $p \supset q$, the antecedent p is a sufficient condition of the consequent q, and the consequent q is a necessary condition of the antecedent p.

Notice that $p \supset q$ if and only if $\neg q \supset \neg p$.

For example: "If Socks is a cat, then Socks is a mammal". Being a cat is a sufficient condition of being a mammal. Being a mammal is a necessary condition of being a cat.

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Model Checking

Given T and M, check whether $M \vDash T$.

For example, using the truth table method we can determine whether $(\neg p \vee q) \wedge (q \supset \neg r \wedge \neg p) \wedge (p \vee r)$ is a model for p = T, q = F, r = T or p = F, q = F, r = F.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \supset \neg r \wedge \neg p$	$p \vee r$	A
T	F	T	F	F	T	T	F
F	F	F	F	T	T	F	F

Observation: useful for checking properties (T) of existing (artificial or natural) systems (M).

Satisfiability

Given T, check whether there exists M such that $M \vDash T$.

For example, using the truth table method we can determine if $(\neg p \vee q) \wedge (q \supset \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with A) is satisfiable.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \supset \neg r \wedge \neg p$	$p \vee r$	A
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
…	…	…	…	…	…	…	…
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

Observation: The first reasoning problem by excellence! Given a set of requirements (T) find a system which satisfies it (e.g. TSM, scheduling)

Validity

Given T, check whether there for all M we have $M \models T$.

For example, using the truth table method we can determine if $(p \supset q) \vee (p \supset \neg q)$ is a valid formula or not.

p	q	$p \supset q$	$\neg q$	$p \supset \neg q$	$(p \supset q) \vee (p \supset \neg q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Observation: Find whether a property (T) is true in all models (of interest). Useful for theory reformulation (using, e.g., equivalence)

Unsatisfiability

Given T, check whether there is no M such that $M \vDash T$.

For example, using the truth table method we can determine if $\neg((p \supset q) \vee (p \supset \neg q))$ is unsatisfiable or not.

p	q	$p \supset q$	$\neg q$	$p \supset \neg q$	$\neg((p \supset q) \vee (p \supset \neg q))$
T	T	T	F	F	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

Observation: Find whether a property (T) is not realisable. Useful check on the suitability of the representation of reality of a LODE theory (e.g., AI, non monotonic resoning, planning)

Logical Consequence

Given T_1 and T_2, check whether $T_1 \vDash T_2$.

For example, using the truth table method we can determine if $\neg q \vee \neg p$ is a logical consequence of the formula $\neg q$.

p	q	$\neg p$	$\neg q$	$\neg q \vee \neg p$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Whenever $\neg q$ is True, $\neg q \vee \neg p$ is also True, making it a logical consequence of $\neg q$.

Observation: The second reasoning problem by excellence. Compute the consequences of a set of facts. (Look at deduction theorem!). Backward reasoning from goals.

Logical Equivalence

Given T_1 and T_2, check whether $T_1 \vDash T_2$ and $T_1 \vDash T_2$.

For example, using the truth table method we can determine whether $p \supset (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \wedge \neg q$	$p \supset (q \wedge \neg q)$	$\neg p$
T	T	F	F	F
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

The truth value is the same for every interpretation, therefore the formulas are logically equivalent.

Observation: Useful to substitute equivalents for equivalents (property reformulation).

Reasoning problems - Correlations

Theorem. If a formula is valid, then it is also satisfiable, and it is also not unsatisfiable. That is:

Validity implies **Satisfiability** implies **not Unsatisfiability**

Theorem. If a formula is unsatisfiable, then it is also not satisfiable, and also not valid. That is:

Unsatisfiability implies **not Satisfiable** implies **not Valid**

Reasoning problems - Correlations

Theorem. The validity, satisfiability and unsatisfiability of a formula and of its negation correlate as follows:

If A is	then ¬ A is
Valid	**Unsatisfiable**
Satisfiable	**Not Valid**
Not Valid	**Satisfiable**
Unsatisfiable	**Valid**

Reasoning problems - Correlations

- **Model checking (= entailment) (MC)** is the core decision problem
- **Satisfiability (SAT)** reduces to generating all models and then test MC
- **Unsatisfiability (UNSAT)** reduces to failure in proving SAT
- **Validity (VAL)** can be reduced to the unsatisfiability of the negation of the input theory
- **Logical Consequence (LC).** Two possibilities
  - Use the deduction theorem to reduce LC to a VAL problem
  - Use the refutation principle to reduc to an UNSAT problem
- **Logical Equivalence (LE)** reduces to LC

Reasoning Problems - observations

Observation 1. Differently from Satisfiability, testing the holding of Validity or Unsatisfiability requires checking all the 2^n interpretations for success. With satisfiability this is only a worst case analysis (only one model, which is also the last to be selected).

Observation 2. For any finite set of formulas Γ, (i.e., $\Gamma = A_1, \dots, A_n$ for some n ≥ 1), Γ is valid (respectively, satisfiable and unsatisfiable) if and only if $A_1 \wedge \dots \wedge A_n$ (respectively, satisfiable and unsatisfiable)

Observation 3. All mainstream reasoning algorithms implement SAT and, to a lesser extent, UNSAT, plus problem reduction.

Example: Valid, Satisfiable or Unsatisfiable?

Prove that

- Blue Fomulas are valid,
- Magenta Formulas are satisfiable but not valid
- Red Formulas are unsatisfiable.

Satisfiable / Valid:

$A \supset A$

$A \lor \neg A$

$\neg \neg A \equiv A$

$\neg (A \land \neg A)$

$A \land B \supset A$

$A \supset A \lor B$

Satisfiable / Not Valid:

$A \lor B$

$A \supset B$

$\neg (A \lor B) \supset C$

Unsatisfiable / Not Valid:

$A \land \neg A$

$\neg (A \supset A)$

$A \equiv \neg A$

$\neg (A \equiv A)$

LoP- The Logic of Propositions

Reasoning about what is True and what is False