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$\mathcal{LOP}$ - The Logic of propositions

Handouts

December 15, 2023

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Chapter 1
$\mathcal{LOP}$ A Logic of Propositions

1.1 Introduction

In the chapter on $\mathcal{LODE}$ we provided the definition below.

Definition 1.1 (Knowledge Bases (Reprise)) [1] The original use of the term knowledge base was to describe one of the two sub-systems of an expert system (a knowledge-based system). A knowledge-based system [*] consists of:

- a knowledge-base representing facts about the world and
- ways of reasoning about those facts to deduce new facts or highlight inconsistencies. ($\mathcal{LOP}$)

$\mathcal{LODE}$ provides the formalization of knowledge Graphs (an evolution of Knowledge bases). In $\mathcal{LOP}$ we concentrate on the formalization of reasoning, namely the second element of knowledge-based systems.

Observation 1.1 (Facts, Assertions, Definitions, (Populated) Descriptions, Propositions (Reprise)) Based on what delivered so far we have the following:

- We depict the world as a set of facts (Set, domain, model, data and knowledge level depictions of the world)
- We structure facts in terms of entities, types, properties (data or knowledge level depictions of the world)
- We describe facts (involving entities, types, properties) in the world using assertions ($\mathcal{LOE}$, language, theory, data level atomic assertions, descriptions of the world)
- We define and inter-relate the concepts (i.e., the meaning of the words) we use in assertions using definitions. This allows us to describe facts at different levels of abstraction ($\mathcal{LOD}$, definitions, knowledge level complex formulas, descriptions of the world)

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[1] Hayes-Roth, Frederick; Donald Waterman; Douglas Lenat (1983). Building Expert Systems. Addison-Wesley.

- We describe the diversity/ variability of concepts using descriptions ($\mathcal{LOD}$, descriptions, knowledge level complex formulas)
- We describe the diversity/ variability of entities populating concepts using grounded descriptions ($\mathcal{LODE}$, data level complex formulas)
- We reason about grounded descriptions using propositions ($\mathcal{LOP}$, truth level complex formulas)

In the literature we have multiple notions of proposition,

Notion 1.1 (Propositions - Google/ Oxford Languages) A proposition is an assertion that expresses a judgement or opinion.

Notion 1.2 (Propositions - Aristotle) A proposition is a sentence which affirms or denies a predicate of a subject.

Notion 1.3 (Propositions - $\mathcal{LOP}$) A proposition is a formula which can be either true or false; it must be one or the other (Law of excluded middle), and it cannot be both (Law of noncontradiction).

We will focus on the latter notion.

Example 1.1 (Propositions) Representing Truth/Falsity is the key for implementing reasoning.

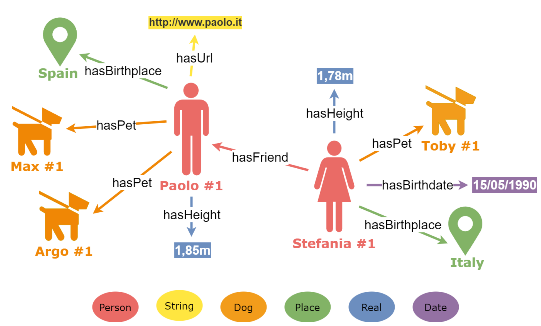


Fig. 1.1 Example of an EG.

Which of the following assertions are intuitively true?

- HasFriend(Paolo#1,Stefania#1)
- Hasheight(Stefania#1, 2m)
- HasPet (Stefania#1, Fido#1)
- Not HasHeight(Stefania#1, 2m)
- HasFriend(Paolo#1,Stefania#1) and HasHeight(Stefania#1, 2m)
- HasFriend(Paolo#1,Stefania#1) or HasHeight(Stefania#1, 2m)
- HF

- HF and HH
- HF or HH
- ...

Select an Interpretation function (the obvious one)? Based on the selected Interpretation function decide which propositions represented in the KG above are intuitively True, False or unknown.

1.2 Domain

Definition 1.2 (Reminder Definition Domain $\mathtt{D}$) The domain can be formalized as the set of etypes, dtypes, a set of classes and the set of relations (as with $\mathcal{LOE}$, $\mathcal{LOD}$, $\mathcal{LODE}$).

$$\mathtt{D} =< \mathtt{E}, \{\mathtt{C}\}, \{\mathtt{R}\} > \tag{1.1}$$

with

$$\begin{aligned} \mathtt{E} &= \{e\} \cup \{v\} \\ \{\mathtt{C}\} &= \mathtt{ET} \cup \mathtt{DT} \cup \mathtt{DET} \\ \{\mathtt{R}\} &= \{\mathtt{OR}\} \cup \{\mathtt{DR}\} \end{aligned}$$

where $\mathtt{E}$ is a set of entities and values, $\mathtt{ET} = \{\mathtt{E_T}\}$, $\mathtt{E_T} = \{e\}$ and $\mathtt{DT} = \{\mathtt{D_T}\}$, $\mathtt{D_T} = \{v\}$ are **set of etypes** and **dtypes**, respectively, and $\mathtt{OR}, \mathtt{DR}$ are **(binary) object** and **data relations**.

1.3 Language

Definition 1.3 (The language $\mathcal{L}^e$)

$$\mathcal{L}^e = \mathcal{L}^e_a \cup \mathcal{L}^e_c \tag{1.2}$$

Definition 1.4 (The language of atomic formulas $\mathcal{L}^e_a$) In this logic, the language of sentences is defined as follows:

$$\mathcal{L}^e_a =< \mathcal{A}_a, \emptyset > \tag{1.3}$$

Observation 1.2 There are no formation rules for atomic formulas. Propositions are judgements about facts, without references to their internal structure, i.e., the entities and relations that compose them. The only goal in $\mathcal{LOP}$ is to reason about truth!

Definition 1.5 (Alphabet $\mathcal{A}_a$) The alphabet is formed by the set of sentences that denote facts in the world. These sentences are called "propositions" in $\mathcal{LOP}$.

$$\mathcal{A}_a =< \{P\} > \tag{1.4}$$

Definition 1.6 (Language of Complex Formulas $\mathcal{L}_c$) The language of complex formulas includes the language of atomic formulas and the rules of complex formula formation, which are highly significant in this logic:

$$\mathcal{L}_c = < \mathcal{L}_a, \mathcal{W}_c > \tag{1.5}$$

Definition 1.7 (Formation Rules of Complex Formulas)

<cwff> ::= <proposition>
<cwff> ::= $\neg$ <cwff> |
::= <cwff> $\wedge$ <cwff> |
::= <cwff> $\vee$ <cwff> |
::= <cwff> $\supset$ <cwff> |
::= <cwff> $\equiv$ <cwff>
<proposition> ::= $\mathcal{P}_1 \ldots \mathcal{P}_n \in \{P\}$

Observation 1.3 (Well-formed formula) As you may notice, a well-formed formula is nothing more than either a single proposition describing a fact or the combination of these prepositions by means of special logical symbols: negation ($\neg$), and ($\wedge$), or ($\vee$), implication ($\supset$) and equivalence ($\equiv$), fundamental for thinking about the world.

1.4 Interpretation function

Definition 1.8 ($\mathcal{LOP}$ Interpretation function) Let $\mathtt{D} =< \mathtt{E}, \{\mathtt{C}\}, \{\mathtt{R}\} >$ be a $\mathcal{LODE}$ domain of interpretation. Let $\mathcal{L}_{\mathcal{LODE}} = \mathcal{L}_{a\mathcal{LODE}} \cup \mathcal{L}_{c\mathcal{LODE}}$ be a $\mathcal{LODE}$ language for D. Let $\mathcal{I}_{\mathcal{LODE}}$ be a $\mathcal{LODE}$ interpretation function, with $\mathcal{I}_{\mathcal{LODE}}\colon \mathcal{L}_{a\mathcal{LODE}} \longrightarrow D$.

Let $\mathcal{L}_{\mathcal{LOP}} = \mathcal{L}_{a\mathcal{LOP}} \cup \mathcal{L}_{c\mathcal{LOP}}$. Let $\mathcal{P} \in \{P\}$ be a $\mathcal{LOP}$ proposition, with $\{P\} = \mathcal{L}_{a\mathcal{LOP}}$. Let $\mathcal{I}_{\mathcal{LOP}}$ be a $\mathcal{LOP}$ interpretation function, with $\mathcal{I}_{\mathcal{LOP}}\colon \{P\} \longrightarrow$ T,F.

Let Translate a bijective (injective and surjective) function such that:

$$\text{for all } A \in \mathcal{L}_{a\mathcal{LODE}}, \text{Translate}(A) = \mathcal{P}_A \text{ with } \mathcal{P}_A \in \{P\}$$

Then we have the following:

- $\mathcal{I}_{\mathcal{LOP}}$ $(\mathcal{P}) = \mathtt{T}$ if and only if $\mathcal{I}_{\mathcal{LODE}}(\text{Translate}^{-1}(\mathcal{P})) \in \mathtt{M}$ ($\Longleftrightarrow$ $\mathtt{M} \models_{\mathcal{LODE}}$ $\text{Translate}^{-1}(\mathcal{P})$)
- $\mathcal{I}_{\mathcal{LOP}}$ $(\mathcal{P}) = \mathrm{F}$ if and only if $\mathcal{I}_{\mathcal{LODE}}(\text{Translate}^{-1}(\mathcal{P})) \notin \mathtt{M}$

Observation 1.4 In the following we drop the name of the logic as not needed. We will in fact focus on $\mathcal{LOP}$.

Example 1.2 (Which truth values of which propositions) Take the example above.

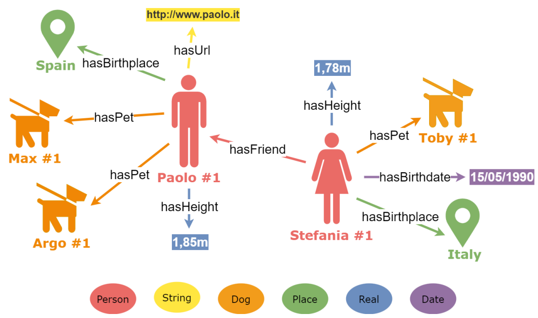


Fig. 1.2 Example of an EG.

- HasFriend(Paolo#1,Stefania#1)
- Hasheight(Stefania#1, 2m)
- HasPet (Stefania#1, Fido#1)
- Not HasHeight(Stefania#1, 2m)
- HasFriend(Paolo#1,Stefania#1) and HasHeight(Stefania#1, 2m)
- HasFriend(Paolo#1,Stefania#1) or HasHeight(Stefania#1, 2m)
- ...

Follow Definition 1.8 and formalize as being True or false the propositions identified as such in the example above.

Definition 1.9 (Theory) A $\mathcal{LOP}$ theory is a set of formulas $w \in \mathcal{T} \subseteq \mathcal{L}$ (as before).

Definition 1.10 (Model) A $\mathcal{LOP}$ model is a set of propositions $\{\mathcal{P}_{\mathcal{T}}\}, \subseteq \{P\}$, with with $\{P\} = \mathcal{L}_a$.

Observation 1.5 (Model) A $\mathcal{LOP}$ model is the set of propositions $\mathcal{P} \in \{P\}$ such that $\mathcal{I}(\mathcal{P})$ = True, namely the set of propositions corresponding to assertions which are true in the corresponding $\mathcal{LODE}$ model. As such, a $\mathcal{LOP}$ model implicitly assigns False to the remaining propositions.

Observation 1.6 (Model) A $\mathcal{LOP}$ model assigns to any proposition either True or False. For any $\mathcal{LODE}$ model the corresponding $\mathcal{LOP}$ model is obtained by adding negative propositions for those assertions do not hold in the $\mathcal{LODE}$ Model.

Observation 1.7 (Model and Interpretation) In $\mathcal{LOP}$, the notion of model and that of interpretation are collapsed. It is said that "... a model is an interpretation which makes true all the formulas in a theory $\mathcal{T}$".

1.5 Entailment Relation

Definition 1.11 (Entailment $\models$) A proposition P is entailed by an interpretation function $\mathcal{I}$ if the facts below are valid:

$$
\begin{array}{llcl}
\mathcal{I} \models P, & \text{if, } \mathcal{I}(P) = True, & \text{with} & P \in \mathcal{P} \\
\mathcal{I} \models \neg P, & \text{if,} & \text{not} & \mathcal{I} \models P \\
\mathcal{I} \models P_1 \wedge P_2, & \text{if } \mathcal{I} \models P_1 & \text{and} & \mathcal{I} \models P_2 \\
\mathcal{I} \models P_1 \vee P_2, & \text{if, } \mathcal{I} \models P_1 & \text{or} & \mathcal{I} \models P_2 \\
\mathcal{I} \models P_1 \supset P_2, & \text{if, when } \mathcal{I} \models P_1, & \text{then} & \mathcal{I} \models P_2 \\
\mathcal{I} \models P_1 \equiv P_2, & \text{if, } \mathcal{I} \models P_1 & \text{if and only if} & \mathcal{I} \models P_2
\end{array}
$$

Definition 1.12 (Entailment, truth, satisfiability) The following statements are equivalent natural language enunciations of the statement $I \models A$:

- the interpretation function (model) I entails the formula A,
- the formula A is true in the interpretation function (model) I,
- the formula A is satisfied by the interpretation function (model) I.

Observation 1.8 (How connectives operate)

¬ *True*	*False*
¬ *False*	*True*

True ∧ *True*	*True*
True ∧ *False*	*False*
False ∧ *True*	*False*
False ∧ *False*	*False*

True ∨ *True*	*True*
True ∨ *False*	*True*
False ∨ *True*	*True*
False ∨ *False*	*False*

True ⊃ *True*	*True*
True ⊃ *False*	*False*
False ⊃ *True*	*True*
False ⊃ *False*	*True*

True ≡ *True*	*True*
True ≡ *False*	*False*
False ≡ *True*	*False*
False ≡ *False*	*True*

Observation 1.9 (The law of the excluded middle)

- The law of the excluded middle "$A \vee \neg A$" is True in all models, for all formulas A.
- All formulas of the above form, independently of the shape of A, are called tautologies.
- Sometimes tautologies are written as T (for truth, as represented in the language)
- The interpretation of T is True

Observation 1.10 (The law of the noncontradiction)

- The law of noncontradiction "$A \wedge \neg A$" is False in all models (that is, True in no model), for all formulas A.

- All formulas of the above form, independently of the shape of A, are called contradictions.
- Sometimes they are written as $\bot$ (for falsity, as represented in the language). Not to be confused with $\bot$ (bottom) in $\mathcal{LOE}$!
- The interpretation of $\bot$ is False

Observation 1.11 (Basic Facts about conjunction/disjunction)

- Same proposition
  - $A \land A \equiv A$
  - $A \lor A \equiv A$
- Commutativity
  - $A \land B \equiv B \land A$
  - $A \lor B \equiv B \lor A$
- De Morgan laws
  - $\neg(A \lor B) \equiv \neg A \land \neg B$
  - $\neg(A \land B) \equiv \neg A \lor \neg B$
- Associativity
  - $(A \land B) \land C \equiv A \land (B \land C)$
  - $(A \lor B) \lor C \equiv A \lor (B \lor C)$
- Distributivity
  - $A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$
  - $A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$

Observation 1.12 (Basic Facts about implication/equivalence)

- Implication and disjunction
  - $(A \supset B) \equiv (\neg A \lor B)$
- Implication and contradiction
  - $\bot \supset A$, for any A
- Equivalence and exclusive disjunction (exor)
  - $(A \equiv B) \equiv \neg(A + B)$
- Implication and negation
  - $A \supset B \equiv \neg B \supset \neg A$
- Implication and equivalence
  - $(A \equiv B) \equiv ((A \supset B) \land (B \supset A))$
- Exclusive and inclusive disjunction

– $(A + B) \equiv (\neg A \wedge B) \vee (\neg B \wedge A)$

Observation 1.13 (Basic Facts about implication/conj/disj)

- Implication and conjunction (1)
  – $(A \wedge B) \supset C \equiv (A \supset C) \vee (B \supset C)$
- Implication and conjunction (2)
  – $(A \wedge B) \supset C \equiv A \supset (B \supset C)$
- Implication and conjunction (3)
  – $(A \wedge B) \supset C \equiv A \supset (\neg B \vee C)$
- Implication and conjunction (4)
  – $A \supset (B \wedge C) \equiv (A \supset B) \wedge (A \supset C)$
- Implication and disjunction (1)
  – $(A \supset (B \vee C) \equiv (A \supset B) \vee (A \supset C)$
- Implication and disjunction (2)
  – $(A \vee B) \supset C \equiv (A \supset B) \wedge (B \supset C)$

Observation 1.14 (Set theoretic representation of an interpretation function) An interpretation $\mathcal{I}$ can be represented set theoretically as the set of propositions that are true in $\mathcal{I}$ (as from the corresponding $\mathcal{LODE}$ theory/model).

Example 1.3 (Set theoretic representation of an interpretation function) Assume we have any formula which contains three atomic propositions p, q, r.

	p	q	r	*Set theoretic representation*
I_1	$True$	$True$	$True$	$\{p, q, r\}$
I_2	$True$	$True$	$False$	$\{p, q\}$
I_3	$True$	$False$	$True$	$\{p, r\}$
I_4	$True$	$False$	$False$	$\{p\}$
I_5	$False$	$True$	$True$	$\{q, r\}$
I_6	$False$	$True$	$False$	$\{q\}$
I_7	$False$	$False$	$True$	$\{r\}$
I_8	$False$	$False$	$False$	$\{\}$

Observation 1.15 (Interpretation and set of propositions) If $\{P\}$ is the set of propositions, an interpretation $\mathcal{I}$ can be thought of as a subset $S \subseteq \{P\}$, and $\mathcal{I}$ is the characteristics function of S, that is

$$A \in S \quad \text{if and only if} \quad \mathcal{I}(A) = \text{True}$$

Example 1.4 Let $\{P\} = \{P, Q\}$. $I(P) = True$ and $I(Q) = False$ can be also expressed with $I = \{P\}$

Observation 1.16 (Number of interpretations) If $|\{P\}|$ is the cardinality of $\{P\}$ then there are $2^{|\{P\}|}$ different interpretations, corresponding to all the different subsets of $\{P\}$.

Observation 1.17 (Models of a theory) A theory $\mathcal{T}$ has usually multiple models. To have only model, a theory $\mathcal{T}$ must be complete, i.e., it must assign a truth value to all propositions allowed by the language. For instance, $\mathcal{T} = \{\mathcal{P}_1 \wedge \neg\mathcal{P}_2\}$, with $\{\mathcal{P}\} = \{\mathcal{P}_1, \mathcal{P}_2\}$. A theory which is complete is said to be maximal.

Observation 1.18 (Maximal theories) A Model $\mathtt{M}$ is (partially) described by multiple theories $\mathcal{T}$. There are multiple maximal theories $\mathcal{T}$ which describe it completely. For instance, assume $\{\mathcal{P}\} = \{\mathcal{P}_1, \mathcal{P}_2\}$. If $\mathtt{M} = \{\mathcal{P}_1\}$ then $\mathcal{T}_1 = \{\mathcal{P}_1 \wedge \neg\mathcal{P}_2\}$, $\mathcal{T}_2 = \{\mathcal{P}_1, \neg\mathcal{P}_2\}$ are maximal theories for $\mathtt{M}$.

Observation 1.19 (Partiality of a theory) The more partial a theory $\mathcal{T}$ is, in terms of positive truth values assigned to propositions, the more models it has. For instance, assume $\{\mathcal{P}\} = \{\mathcal{P}_1, \mathcal{P}_2\}$.

- $\mathcal{T} = \{\}$ has four models
- $\mathcal{T} = \{\mathcal{P}_1 \vee \mathcal{P}_2\}$, has three models
- $\mathcal{T} = \{\mathcal{P}_1\}$ has two models
- $\mathcal{T} = \{\mathcal{P}_1 \wedge \mathcal{P}_2\}$ has one model

The key point is that an increase in $\mathcal{LODE}$ partiality is modeled by increase of the number of $\mathcal{LOP}$ models

Observation 1.20 (Minimal models) Given a theory $\mathcal{T}$, there is no minimal Model M which is the intersection of all models of T, the main cause being disjunction. For instance, assume $\{\mathcal{P}\} = \{\mathcal{P}_1, \mathcal{P}_2\}$. $\mathcal{T} = \{\mathcal{P}_1 \vee \mathcal{P}_2\}$ has four models and no minimal model.

Observation 1.21 ($\mathcal{LODE}$ minimal models) $\mathcal{LODE}$ theories have minimal models (they contain no disjunctions, they have single premises)

Proposition 1.1 (Interpretation equivalence with respect to a formula) *If for all and only the atomic propositions P occurring in a formula A we have*

$$I(P) = I'(P),$$

then

$$I \models A \quad iff \quad I' \models A.$$

Observation 1.22 (Interpretation equivalence with respect to a formula) The proposition above must be read as follows

- the truth value of atomic propositions which occur in A fully determines the truth value of A;
- The truth value of the atomic propositions which do not occur in A play no role in the computation of the truth value of A;

1.5.1 Entailment Properties

Proposition 1.2 (Deduction theorem - Logical consequence, validity)

$$\Gamma, \phi \models \psi \quad \textit{if and only if } \Gamma \models \phi \supset \psi$$

Observation 1.23 (Deduction theorem - Logical consequence, validity)

- The deduction theorem explains (left to right) the meaning of implication. Implication is how we express logical consequence in language.
- It also says (right to left) that from absurdity (i.e, $\mathcal{P} \wedge \neg\mathcal{P}$),we can derive everything, any formula (and assertion) A.

Proposition 1.3 (Refutation principle - Logical consequence, unsatisfiability)

$$\Gamma \models \phi \quad \textit{if and only if } \Gamma \cup \{\neg\phi\} \textit{ is unsatisfiable}$$

Observation 1.24 (Refutation principle - Logical consequence, unsatisfiability)

- The refutation principle explains the meaning of negation. It captures the fact that absurdity (i.e, $\mathcal{P} \wedge \neg\mathcal{P}$) cannot be satisfied by any model depicting facts in the real world.
- Algorithmitically, it suggests how to reason backwards from goals.

1.6 Modeling Mistakes

Observation 1.25 We express conjunction with many words other than "and", including "but," "moreover," "however, "although", and "even though".

For example: "I enjoyed the holiday, even though it rained a lot" can be translated into the facts "I enjoyed the holiday" and "It rained a lot".

Observation 1.26 Sometimes "and" does not join whole propositions into a compound proposition. Sometimes it simply joins nouns. This cannot be paraphrased. In these cases, the "and" is expressed inside the propositional variable, and not as logical connective.

For example: "Bert and Ernie are brothers". This cannot be paraphrased. "Bert is a brother and Ernie is a brother", for that does not assert that they are brothers to each other.

Observation 1.27 Sometimes "and" joins adjectives.

For example: "The leech was long and wet and slimy." This can be paraphrased as "The leech was long, and the leech was wet, and the leech was slimy."

Observation 1.28 (Inclusive vs. exclusive disjunction) The natural, but long-winded, way to express exclusive disjunction is $(\neg p \wedge q) \vee (p \wedge \neg q)$. The way to say they have different truth values is to deny their equivalence: $\neg(p \equiv q)$.

For example: When a menu says "cream or sugar", it uses an inclusive "or", because you may take one, the other, or both. But when it says "coffee or tea", it uses an exclusive "or", because you are not invited to take both.

Observation 1.29 (Implication) $p \supset q$ translates a wide variety of English expressions, for example, "if p, then q", "if p, q", "p implies q", "p entails q", "p therefore q", "p hence q", "q if p", "q provided p", "q follows from p", "p is the sufficient condition of q", and "q is the necessary condition of p". The least intuitive is "p only if q" (to be understood from $\neg q \supset \neg p$).

For example the following all translates to $p \supset q$:

- If Mario goes to the party, (then) I'll go too.
- I'll go to the party if/provided that Mario comes too.
- I'll go to the party only if Mario goes
- Mario going to the party is the sufficient condition of me going to the party.
- Me going to the party is necessary condition of Mario going to the party.
- The decrease in white blood cells implies the antibiotic is working.
- The solution of one problem would immediately entail the solution of all others which belonged to the same class.

Observation 1.30 (Even if) "p even if q" means "p whether or not q" or "p regardless of q". Therefore one perfectly acceptable translation of it is simply "p". If you want to spell out the claim of "regardlessness", then you could write "$p \wedge (q \vee \neg q)$".

For example:

- I'll go to the party even if Mario doesn't go.
- I'll go to the party whether or not Mario goes.
- I'll go to the party regardless of whether Mario comes or not.

Observation 1.31 (Unless) Sometimes "unless" should be translated as inclusive disjunction, and sometimes as exclusive disjunction.

For example (**inclusive disjunction**): "I'll go to the party unless I get another offer" means that I'll go if nothing else comes along. In many contexts it also means that I might go anyway; the second offer might be worse. So I'll go or I'll get another offer or both.

For example (**exclusive disjunction**): Consider by contrast, "I'll go to the party unless Rufus is there". In many contexts this means that if I learn Rufus is going, then I'll change my mind and not go. So either I'll go or Rufus will go but not both.

Observation 1.32 (Necessary and Sufficient condition)

- We say that p is a sufficient condition of q when p's truth guarantees q's truth. By contrast, q is a necessary condition of p when q's falsehood guarantees p's falsehood.
- In the ordinary material implication, $p \supset q$, the antecedent p is a sufficient condition of the consequent q, and the consequent q is a necessary condition of the antecedent p.
- Notice that $p \supset q$ if and only if $\neg q \supset \neg p$.

For example: "If Socks is a cat, then Socks is a mammal". Being a cat is a sufficient condition of being a mammal. Being a mammal is a necessary condition of being a cat.

1.7 Reasoning Problems

Now that we have all the necessary notions, we can address and reason about some common problems that must be solved when we want to make judgments about the world:

Reasoning Problem 1.1 (Model checking) Given $\mathcal{T}$ and $\mathtt{M}$, check whether $\mathtt{M} \models \mathcal{T}$.

Example 1.5 (Model Checking) Use the truth table method to determine whether $(\neg p \vee q) \wedge (q \supset \neg r \wedge \neg p) \wedge (p \vee r)$ is a model for p = T, q = F, r = T or p = F, q = F, r = F.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \supset \neg r \wedge \neg p$	$(p \vee r)$	A
T	F	T	F	F	T	T	F
F	F	F	T	T	T	F	F

Observation 1.33 (Using model checking) useful for checking properties (T) of existing (artificial or natural) systems (M).

Reasoning Problem 1.2 (Satisfiability) Given $\mathcal{T}$, check whether there exists $\mathtt{M}$ such that $\mathtt{M} \models \mathcal{T}$.

Example 1.6 (Satisfiability) Use the truth table method to determine whether $(\neg p \vee q) \wedge (q \supset \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with A) is satisfiable.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \supset \neg r \wedge \neg p$	$(p \vee r)$	A
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	T	T	F	F	T	F
F	F	T	T	F	T	T	T
F	F	F	T	T	T	F	F

There exists an interpretation satisfying A, thus A is satisfiable.

Observation 1.34 (Using satisfiability) The first reasoning problem by excellence! Given a set of requirements (T) find a system which satisfies it (e.g. TSM, scheduling)

Reasoning Problem 1.3 (Validity) Given $\mathcal{T}$, check whether there for all $\mathtt{M}$ we have $\mathtt{M} \models \mathcal{T}$.

Example 1.7 (Validity) Compute the truth table of $(p \supset q) \vee (p \supset \neg q)$ to see whether this formula is valid or not

p	q	$p \supset q$	$\neg q$	$p \supset \neg q$	$(p \supset q) \vee (p \supset \neg q)$
T	T	T	F	F	$\boldsymbol{T}$
T	F	F	T	T	$\boldsymbol{T}$
F	T	T	F	T	$\boldsymbol{T}$
F	F	T	T	T	$\boldsymbol{T}$

The formula is valid since it is satisfied by every interpretation.

Observation 1.35 (Using Validity) Find whether a property (or theorey) is true in all models (of interest). Useful for theory reformulation (using, e.g., equivalence)

Reasoning Problem 1.4 (Unsatisfiability) Given $\mathcal{T}$, check whether there is no $\mathtt{M}$ such that $\mathtt{M} \models \mathcal{T}$.

Example 1.8 (Unsatisfiability) Compute the truth table of $\neg((p \supset q) \vee (p \supset \neg q))$ to see whether this formula is unsatsifiable or not

p	q	$p \supset q$	$\neg q$	$p \supset \neg q$	$\neg((p \supset q) \vee (p \supset \neg q))$
T	T	T	F	F	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

The formula is unsatisfiable since there is no interpretation that satisfies it.

Observation 1.36 (Using Unsatisfiability) Find whether a property or theory is not satisfiable. Useful to check on the suitability of the representation of reality of a LODE theory (e.g., AI, non monotonic reasoning, planning). A theory or property which is not satisfiable cannot describe reality

Reasoning Problem 1.5 (Logical consequence) Given $\mathcal{T}_1$ and $\mathcal{T}_2$, check whether $T_1 \models T_2$.

Example 1.9 (Logical consequence) Use the truth table method to determine whether $\neg p$ is a logical consequence of the formula A=$\{q \vee r \wedge q \supset \neg p \wedge \neg(r \wedge p)\}$

p	q	r	$q \vee r$	$q \supset \neg p$	$\neg(r \wedge p)$	A	$\neg p$
T	T	T	T	F	F	F	F
T	T	F	T	F	T	F	F
T	F	T	T	T	F	F	F
T	F	F	F	T	T	F	F
F	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	F	T

Observation 1.37 (Logical consequence) Whenever the premise holds then the conclusion must hold. The truth value of teh conclusion is irrelevant when the premise does not hold

Observation 1.38 (Using Logical consequence) The second reasoning problem by excellence. Compute the consequences of a set of facts. Backward reasoning from goals. Because of the deduction theorem can be directly encoded into a satisfiability problem.

Reasoning Problem 1.6 (Logical equivalence) Given $\mathcal{T}_1$ and $\mathcal{T}_2$, check whether $T_1 \models T_2$ and $T_2 \models T_1$.

Example 1.10 (Logical equivalence) Use the truth tables method to determine whether $p \supset (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \wedge \neg q$	$p \supset (q \wedge \neg q)$	$\neg p$
T	T	F	F	F
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Observation 1.39 (Logical equivalence) The holding/ not holding of premise and conclusion must coincide for all interpretations.

Observation 1.40 (Using Logical consequence) Useful to substitute equivalents for equivalents (property reformulation).

Proposition 1.4 *If a formula is valid, then it is also satisfiable, and it is also not unsatisfiable. That is:*

$$\textit{Validity} \implies \textit{Satisfiability} \implies \textit{not Unsatisfiability}$$

Proposition 1.5 *If a formula is unsatisfiable, then it is also not satisfiable, and also not valid. That is:*

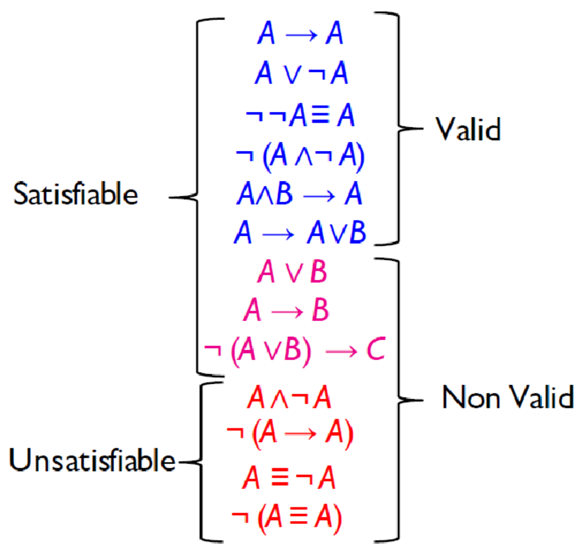
$$\textit{Unsatisfiability} \implies \textit{not Satisfiable} \implies \textit{not Valid}$$

The validity, satisfiablity and unsatisfiabolity of a formula and of its negation are correlated as follows:

Proposition 1.6

if A is	*then ¬ A is*
Valid	*Unsatisfiable*
Satisfiable	*not Valid*
not Valid	*Satisfiable*
Unsatisfiable	*Valid*

Example 1.11 Prove that the blue formulas are valid, that the magenta formulas are satisfiable but not valid, and that the red formulas are unsatisfiable.



Observation 1.41 (Correlations)

- Model checking (= entailment) (MC) is the core decision problem
- Satisfiability (SAT) reduces to generating all models and then test MC
- Unsatisfiability (UNSAT) reduces to failure in proving SAT

- Validity (VAL) can be reduced to the unsatisfiability of the negation of the input theory
- Logical Consequence (LC)
  - Can be reduced to SAT via the deduction theorem
  - Can be reduced to UNSAT via the refutation principle
- Logical Equivalence (LE) reduces to LC

Observation 1.42 (The practice of reasoning algorithms) All mainstream reasoning algorithms implement SAT and, to a lesser extent, UNSAT, plus problem reduction.

Observation 1.43 (Complexity of reasoning) Differently from satisfiability, checking the holding of validity a and unsatisfiability require checking all the 2^n interpretations. With satisfiability, this is only a worst case analysis as a satisfiability algorithm will stop as soon as the first model is found.

Observation 1.44 (Satisfiability of set of formulas or theory) For any finite set of formulas Γ, (i.e., $\Gamma = \{A_1, \ldots, A_n\}$ for some $n \geq 1$), Γ is valid (respectively satisfiable and unsatisfiable) if and only if $A_1 \wedge \ldots \wedge A_n$ (respectively satisfiable and unsatisfiable).

1.8 Exercises

1.8.1 Basic concepts

Exercise 1.1 Which of the following are well formed propositional formulas?

1. $\vee pq$
2. $(\neg(p \supset (q \wedge p)))$
3. $(\neg(p \supset (q = p)))$
4. $(\neg(\diamond(q \vee p)))$
5. $(p \wedge \neg q) \vee (q \supset r)$
6. $p\neg r$

Exercise 1.2 Let's consider the interpretation v where $v(p) = \mathrm{F}$, $v(q) = \mathrm{T}$, $v(r) = \mathrm{T}$. Does v satisfy the following propositional formulas?

1. $(p \supset \neg q) \vee \neg(r \wedge q)$
2. $(\neg p \vee \neg q) \supset (p \vee \neg r)$
3. $\neg(\neg p \supset \neg q) \wedge r$
4. $\neg(\neg p \supset q \wedge \neg r)$

Exercise 1.3 Consider that we have a propositional language L and a theory T written in the language L. Let be I(L) an interpretation of L: the set of atomic propositions and negated atomic propositions that are true. Indicate which of the following statements are true:

1. An interpretation I(L) is such that for every formula A belonging to L we have A belonging to I(L) or ¬ A belonging to I(L).
2. A theory T is such that for every formula A belonging to L we have A belonging to T or ¬ A belonging to T.
3. A model M is an interpretation of a language L that satisfies all the formulas of T.
4. Every model M of a consistent theory T is such that T is a proper or improper subset of M.
5. An inconsistent theory has only one model.
6. Given a language L with N atomic propositions, a tautology (a valid formula) has 2^N models.

Exercise 1.4 Say which of the following statements are true:

1. If T_1 and T_2 are $\mathcal{LOP}$ theories, then, if $T_1 \models A$ and $T_2 \cup \{A\} \models B$ then $T_1 \cup T_2 \models B$
2. Given a formula A in $\mathcal{LOP}$, the number of subformulas of A is equal to the number of logic and non-logic symbols occuring in A, plus 1. (|subformulas| = $|A|$+1)
3. T is a theory in $\mathcal{LOP}$ if and only if $T \models A$ implies that $A \in T$
4. A $\mathcal{LOP}$ theory always contains an infinite set of formulas.
5. If the $\mathcal{LOP}$ formula A is satisfiable then $\neg A$ is unsatisfiable
6. If T_1 and T_2 are two $\mathcal{LOP}$ theories then, $T_1 \models A$ and $T_1 \cup T_2 \models A$ if and only if T_2 is not empty
7. The set of $\mathcal{LOP}$ formulas $\{A, A \supset B, B\}$ is a $\mathcal{LOP}$ theory.
8. Given a $\mathcal{LOP}$ theory, if there is a finite set of axioms that axiomatizes T, then there is more than one of this set of axioms.

Exercise 1.5 Let A be an atomic proposition in $\mathcal{LOP}$ and B a generic propositional variable. Indicate which of the following answers are correct:

1. ¬ A it is unsatisfiable
2. ¬ A ∨ B is satisfiable
3. (A ∧ B) ⊃ B is satisfiable
4. (A ∧ B) ⊃ is valid
5. (¬ A ∨ A) is satisfiable, but not valid
6. If B is a tautology, then A ⊃ B is valid

1.8.2 Informal to formal

Exercise 1.6 Let's consider a propositional language where

- p means *Paola is happy*,
- q means *Paola paints a picture*,
- r means *Renzo is happy*.

Formalize the following sentences:

1. *if Paola is happy and paints a picture then Renzo isn't happy*

2. *if Paola is happy, then she paints a picture*
3. *Paola is happy only if she paints a picture*

Exercise 1.7 Let's consider a propositional language where

- p means *x is a prime number*,
- q means *x is odd*.

Formalize the following sentences:

1. *x being prime is a sufficient condition for x being odd*
2. *x being odd is a necessary condition for x being prime*

Exercise 1.8 Let A =*Aldo is Italian* and B =*Bob is English*. Formalize the following sentences:

1. *Aldo isn't Italian*
2. *Aldo is Italian while Bob is English*
3. *If Aldo is Italian then Bob is not English*
4. *Aldo is Italian or if Aldo isn't Italian then Bob is English*
5. *Either Aldo is Italian and Bob is English, or neither Aldo is Italian nor Bob is English*

Exercise 1.9 Let's consider a propositional language where

- A =*Angelo comes to the party*,
- B =*Bruno comes to the party*,
- C =*Carlo comes to the party*,
- D =*Davide comes to the party*.

Formalize the following sentences:

1. *If Davide comes to the party then Bruno and Carlo come too*
2. *Carlo comes to the party only if Angelo and Bruno do not come*
3. *Davide comes to the party if and only if Carlo comes and Angelo doesn't come*
4. *If Davide comes to the party, then, if Carlo doesn't come then Angelo comes*
5. *Carlo comes to the party provided that Davide doesn't come, but, if Davide comes, then Bruno doesn't come*
6. *A necessary condition for Angelo coming to the party, is that, if Bruno and Carlo aren't coming, Davide comes*
7. *Angelo, Bruno and Carlo come to the party if and only if Davide doesn't come, but, if neither Angelo nor Bruno come, then Davide comes only if Carlo comes*

Exercise 1.10 Socrate says: *If I'm guilty, I must be punished; I'm guilty. Thus I must be punished.* Is the argument logically correct?

Exercise 1.11 Socrate says: *If I'm guilty, I must be punished; I'm not guilty. Thus I must not be punished.* Is the argument logically correct?

Exercise 1.12 Socrate says: *If I'm guilty, I must be punished; I must not be punished. Thus I'm not guilty.* Is the argument logically correct?

Exercise 1.13 Socrate says: *If I'm guilty, I must be punished; I must be punished. Thus I'm guilty.* Is the argument logically correct?

Exercise 1.14 Angelo, Bruno and Carlo are three students that took the Logic exam. Let's consider a propositional language where

- A =*Aldo passed the exam*,
- B =*Bruno passed the exam*,
- C =*Carlo passed the exam*.

Formalize the following sentences:

1. *Carlo is the only one passing the exam*
2. *Aldo is the only one not passing the exam*
3. *Only one, among Aldo, Bruno and Carlo, passed the exam*
4. *At least one among Aldo, Bruno and Carlo passed*
5. *At least two among Aldo, Bruno and Carlo passed the exam*
6. *At most two among Aldo, Bruno and Carlo passed the exam*
7. *Exactly two, among Aldo, Bruno and Carlo passed the exam*

Exercise 1.15 Let's consider a propositional language where

- A =*Angelo comes to the party*,
- B =*Bruno comes to the party*,
- C =*Carlo comes to the party*,
- D =*Davide comes to the party*.

Formalize the following sentences:

1. *Angelo comes to the party while Bruno doesn't*
2. *Either Carlo comes to the party, or Bruno and Davide don't come*
3. *If Angelo and Bruno come to the party, then Carlo comes provided that Davide doesn't come*
4. *Carlo comes to the party if Bruno and Angelo don't come, or if Davide comes*
5. *If Angelo comes to the party then Bruno or Carlo come too, but if Angelo doesn't come to the party, then Carlo and Davide come*

Exercise 1.16 Formalize the following arguments and verify whether they are correct:

- *If Carlo won the competition, then either Mario came second or Sergio came third. Sergio didn't come third. Thus, if Mario didn't come second, then Carlo didn't win the competition.*
- *If Carlo won the competition, then either Mario came second or Sergio came third. Mario didn't come second. Thus, if Carlo won the competition, then Sergio didn't come third.*
- *If Carlo won the competition, then Mario came second and Sergio came third. Mario didn't come second. Thus Carlo didn't win the competition.*
- *If Carlo won the competition, then, if Mario came second then Sergio came third. Mario didn't come second. Thus, either Carlo won or Sergio arrived third*

- *If you play and you study you'll pass the exams, while if you play and don't study you won't pass. Thus, if you play, either you study and you'll pass the exams, or you don't study and you won't pass.*

Exercise 1.17 Let's consider a propositional language where p stands for "Paola is happy", q stands for "Paola points a picture", and r stands for "Renzo is happy". Formalize the following sentences:

1. "if Paola is happy and points a picture then Renzo isn't happy"
2. "if Paola is happy, then she points a picture"
3. "It is not the case that Paola is happy and she does not point a picture"

Exercise 1.18 1. The requirements of a railway system contains the following description of the behavior of a train a level crossing:

- The safety light of a level crossing can be in only one of these two states: red or green.
- If the level crossing is closed then the traffic light is green.
- The traffic light cannot be green if the level crossing is open.
- The train can move if and only if the traffic light is green

Imagining to use the following formalization:

- OPEN =Level crossing is open
- ¬ OPEN = Level crossing is closed
- GREEN = Traffic light is green
- RED = Traffic light is red
- MOVE = The train is moving
- ¬ MOVE = The train is still

indicate which of the following theories captures the informal representation described above.

a. RED $\Longleftrightarrow$ ¬GREEN
¬ OPEN ⊃ GREEN
OPEN ⊃ RED
GREEN $\Longleftrightarrow$ MOVE

b. RED ⊃ ¬GREEN
OPEN ∧ ¬ RED
GREEN ⊃ MOVE
MOVE ⊃ GREEN

c. RED $\Longleftrightarrow$ GREEN
¬ OPEN ⊃ GREEN
OPEN ⊃ GREEN
GREEN $\Longleftrightarrow$ MOVE

d. RED $\Longleftrightarrow$ ¬ GREEN
¬ OPEN ∨ GREEN
OPEN ⊃ RED
GREEN ⊃ MOVE

MOVE ⊃ GREEN

2. Suppose we want to verify the following condition:

   The train moves with the level crossing open

   How would you formalise this condition? Choose one of this:

   - ¬ (OPEN ∧ MOVE)
   - OPEN ⊃ MOVE
   - OPEN $\iff$ MOVE
   - MOVE ∧ OPEN

Exercise 1.19 We want to represent in $\mathcal{LOP}$ the knowledge necessary for a robot to operate in an environment in which there are three colored blocks: one red, one green and one yellow. Blocks can stand on top of each other or side by side. The robot is only able to move a block if it has no other blocks on it.
We represent that one block is on top of another by using propositional letters in the form COLOR_OF_COLOR. For example: RED_Over_GREEN, if true, indicates that the RED block is on top of the GREEN block.
We also represent the fact that the robot can move a block with the following propositional letter: MOVABLE_BLOCK. For example: RED_MOVABLE, if true, indicates that the RED block can be moved. In other words, the RED block has no blocks above it.
Which of the following formulas are valid in our world?

1. (RED_OVER_GREEN ∧ GREEN_OVER_YELLOW) ⊃ MOVABLE_RED
2. MOVABLE_RED ⊃ (¬ GREEN_OVER_RED ∧¬ YELLOW_OVER_RED)
3. (GREEN_OVER_RED ∨ YELLOW_OVER_RED) ⊃ ¬ MOVABLE_RED
4. MOVABLE_RED $\iff$ RED_OVER_GREEN ∧¬ YELLOW_OVER_RED
5. MOVABLE_RED $\iff$ RED_OVER_GREEN ∧ GREEN_OVER_YELLOW
6. MOVABLE_RED ⊃ RED_OVER_GREEN ∧ GREEN_OVER_YELLOW

1.8.3 Reasoning

Exercise 1.20 Find an interpretation that does not satisfy the following formula:

$$(\neg(A \vee B) \wedge (C \wedge \neg D)) \supset (\neg C \vee B)$$

specifying for each propositional variable the corresponding truth value assigned by the interpretation (True, False, Unassigned)

Exercise 1.21 Compute the truth table of $(F \vee G) \wedge \neg(F \wedge G)$.

Exercise 1.22 Use the truth tables method to determine whether $(p \supset q) \vee (p \supset \neg q)$ is valid.

Exercise 1.23 Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \supset \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with φ) is satisfiable.

Exercise 1.24 Use the truth tables method to determine whether the formula $\varphi : p \wedge \neg q \supset p \wedge q$ is a logical consequence of the formula $\psi : \neg p$.

Exercise 1.25 Use the truth tables method to determine whether $p \supset (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

Exercise 1.26 Compute the truth tables for the following propositional formulas:

- $(p \supset p) \supset p$
- $p \supset (p \supset p)$
- $p \vee q \supset p \wedge q$
- $p \vee (q \wedge r) \supset (p \wedge r) \vee q$
- $p \supset (q \supset p)$
- $(p \wedge \neg q) \vee \neg(p \leftrightarrow q)$

Exercise 1.27 Use the truth table method to verify whether the following formulas are valid, satisfiable or unsatisfiable:

- $(p \supset q) \wedge \neg q \supset \neg p$
- $(p \supset q) \supset (p \supset \neg q)$
- $(p \vee q \supset r) \vee p \vee q$
- $(p \vee q) \wedge (p \supset r \wedge q) \wedge (q \supset \neg r \wedge p)$
- $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$
- $(p \vee q) \wedge (\neg q \wedge \neg p)$
- $(\neg p \supset q) \vee ((p \wedge \neg r) \leftrightarrow q)$
- $(p \supset q) \wedge (p \supset \neg q)$
- $(p \supset (q \vee r)) \vee (r \supset \neg p)$

Exercise 1.28 Use the truth table method to verify whether the following logical consequences and equivalences are correct:

- $(p \supset q) \models \neg p \supset \neg q$
- $(p \supset q) \wedge \neg q \models \neg p$
- $p \supset q \wedge r \models (p \supset q) \supset r$
- $p \vee (\neg q \wedge r) \models q \vee \neg r \supset p$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $(p \vee q) \wedge (\neg p \supset \neg q) \equiv q$
- $(p \wedge q) \vee r \equiv (p \supset \neg q) \supset r$
- $(p \vee q) \wedge (\neg p \supset \neg q) \equiv p$
- $((p \supset q) \supset q) \supset q \equiv p \supset q$

Exercise 1.29 Check if $(P \wedge Q) \vee (R \supset S)$ is satisfiable.

Exercise 1.30 Check if $(P \wedge Q) \vee (R \supset S)$ is valid.

Exercise 1.31 Use the truth table method to determine whether $\neg p$ is a logical consequence of the formula A = $\{q \vee r \wedge q \supset \neg p \wedge \neg(r \wedge p)\}$

Exercise 1.32 Use the truth tables method to determine whether $p \supset (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

1.8.4 Solving problems

Exercise 1.33 (Treasure or Trap) Aladdin finds two trunks A and B in a cave. He knows that each of them either contains a treasure or a fatal trap.

On trunk A is written: *At least one of these two trunks contains a treasure.*

On trunk B is written: *In A there's a fatal trap.*

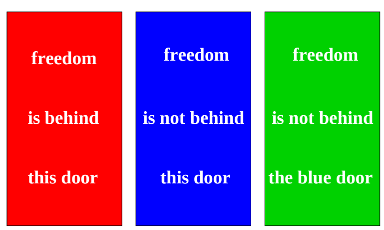
Aladdin knows that either both the inscriptions are true, or they are both false.
Can Aladdin choose a trunk being sure that he will find a treasure? If this is the case, which trunk should he open?

Exercise 1.34 (The three boxes) Three boxes are presented to you. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are:

- Box 1: *The gold is not here*
- Box 2: *The gold is not here*
- Box 3: *The gold is in Box 2*

Only one message is true; the other two are false. Which box has the gold? *Formalize the puzzle in $\mathcal{LOP}$ and find the solution using a truth table.*

Exercise 1.35 (The three doors) Kyle, Neal, and Grant find themselves trapped in a dark and cold dungeon (HOW they arrived there is another story). After a quick search the boys find three doors, the first one red, the second one blue, and the third one green. Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means almost certain death. On each door there is an inscription:



Given the fact that at LEAST ONE of the three statements on the three doors is true and at LEAST ONE of them is false, which door would lead the boys to safety?

Exercise 1.36 (The Labyrinth Guardians.) You are walking in a labyrinth and all of a sudden you find yourself in front of three possible roads: the road on your left is paved with gold, the one in front of you is paved with marble, while the one on your right is made of small stones. Each street is protected by a guardian. You talk to the guardians and this is what they tell you:

- The guardian of the gold street: *This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center.*
- The guardian of the marble street: *Neither the gold nor the stones will take you to the center.*
- The guardian of the stone street: *Follow the gold and you'll reach the center, follow the marble and you will be lost.*

Given that you know that all the guardians are liars, can you choose a road being sure that it will lead you to the center of the labyrinth? If this is the case, which road you choose? *Provide a propositional language and a set of axioms that formalize the problem and show whether you can choose a road being sure it will lead to the center.*

Exercise 1.37 (Binary Strings) Consider the finite set of binary strings

$$\left\{ \begin{array}{l} (000000), (100000), (110000), (111000), (111100), (111110), \\ (111111), (011111), (001111), (000111), (000011), (000001) \end{array} \right\}$$

Explain how it is possible to represent such a set in a propositional formula and find the most compact representation.

Exercise 1.38 (Graph coloring problem) Provide a propositional language and a set of axioms that formalizes the graph coloring problem of a graph with at most n nodes, with connection degree $\leq m$, and with less then $k + 1$ colors.

- *node degree*: number of adjacent nodes
- *connection degree* of a graph: max among all the degree of its nodes
- *graph coloring problem*: given a non-oriented graph, associate a color to each of its nodes in such a way that no pair of adjacent nodes have the same color.

Exercise 1.39 (Traffic light) Define a propositional language which allows to describe the state of a traffic light on different instants. With the language defined above provide a (set of) formulas which expresses the following facts:

1. the traffic light is either green, or red or orange;
2. the traffic light switches from green to orange, from orange to red, and from red to green;
3. it can keep the same color over at most 3 successive states.

Exercise 1.40 (Sudoku) Sudoku is a placement puzzle. The aim of the puzzle is to enter a numeral from 1 through 9 in each cell of a grid, most frequently a 9×9 grid made up of 3×3 subgrids (called "regions"), starting with various numerals given in some cells (the "givens"). Each row, column and region must contain only one instance of each numeral. Its grid layout is like the one shown in the following schema

		9				7		
	4		5		9		1	
3				1				2
	1			6			7	
		2	7		1	8		
	5			4			3	
7				3				4
	8		2		4		6	
		6				5		

Provide a formalization in $\mathcal{LOP}$ of the Sudoku problem, so that any truth assignment to the propositional variables that satisfy the axioms is a solution for the puzzle.

Exercise 1.41 Suppose we know that:

- *if Paolo is thin, then Carlo is not blonde or Roberta is not tall*
- *if Roberta is tall then Sandra is lovely*
- *if Sandra is lovely and Carlo is blonde then Paolo is thin*
- *Carlo is blonde*

Can we deduce that *Roberta is not tall* ?

Exercise 1.42 Anna and Barbara carpool to work. On any day, either Anna drives Barbara or Barbara drives Anna. In the former case, Anna is the driver and Barbara is the passenger; in the latter case Barbara is the driver and Anna is the passenger. Formalize the problem using the following propositions:

1. *Anna drives Barbara*
2. *Barbara drives Anna*
3. *Anna is the driver*
4. *Barbara is the driver*
5. *Anna is the passenger*
6. *Barbara is the passenger*

Solutions

Exercises of Chapter ??

Reduction to Negative Normal Form (NNF)

Solution **??**

1. $\neg(\neg p \vee q) \vee (\neg r \vee \neg s)$
2. $(\neg\neg p \wedge \neg q) \vee (\neg r \vee \neg s)$
3. $(p \wedge \neg q) \vee (\neg r \vee \neg s)$

Solution **??**

1. $\neg(\neg p \rightarrow q) \vee (q \rightarrow \neg r)$
2. $\neg(p \vee q) \vee (\neg q \vee \neg r)$
3. $(\neg p \wedge \neg q) \vee (\neg q \vee \neg r)$

Reduction to Conunctive Normal Form (NNF)

Solution **??**

1. $\neg(\neg p \vee q) \vee (\neg r \vee \neg s)$
2. $(\neg\neg p \wedge \neg q) \vee (\neg r \vee \neg s)$
3. $(p \wedge \neg q) \vee (\neg r \vee \neg s) \quad NNF$
4. $(p \vee \neg r \vee \neg s) \wedge (\neg q \vee \neg r \vee \neg s)$

Solution **??**

1. $\neg(\neg p \rightarrow q) \vee (q \rightarrow \neg r)$
2. $\neg(p \vee q) \vee (\neg q \vee \neg r)$
3. $(\neg p \wedge \neg q) \vee (\neg q \vee \neg r) \quad NNF$
4. $(\neg p \vee \neg q \vee \neg r) \wedge (\neg q \vee \neg r)$

Solution **??**

- $(\neg A \vee B \vee C) \wedge (\neg B \vee A \vee C)$
- $(C \vee A \vee \neg B \vee C) \wedge (B \vee \neg A \vee C)$

Solution **??**

- D,¬ C
- C, A, B
- C, A, D

Solution **??** [Example $(q \wedge p) \vee \neg p$]

$$\begin{aligned}
CNF((q \wedge p) \vee \neg p) \\
CNF((q \wedge p)\,) \otimes CNF(\,\neg p) \\
(CNF(q\,) \wedge CNF(\,p)) \otimes \neg p \\
(q \wedge p) \otimes \neg p \\
(q \vee \neg p) \wedge (q \vee \neg p)
\end{aligned}$$

Solution **??** [Example $(p \supset q) \equiv (\neg q \supset \neg p)$]

$CNF((p \supset q) \equiv (\neg q \supset \neg p))$
$CNF((p \supset q) \supset (\neg q \supset \neg p)\,) \wedge CNF(\,(\neg q \supset \neg p) \supset (p \supset q))$
$CNF((\neg(\mathrm{p} \supset \mathrm{q}))\,) \otimes CNF(\,(\neg \mathrm{q} \supset \neg \mathrm{p})\,) \wedge CNF(\,(\neg(\neg \mathrm{q} \supset \neg \mathrm{p}))\,) \otimes CNF(\,(\mathrm{p} \supset \mathrm{q}))$
$(CNF(\mathrm{p}\,) \wedge CNF(\,\neg q)\,) \otimes (CNF(\mathrm{q}\,) \otimes CNF(\,\neg \mathrm{p})) \wedge (CNF((\neg q)\,) \wedge CNF(\,\mathrm{p})) \otimes$
$(CNF(\neg \mathrm{p}\,) \otimes CNF(\,\mathrm{q}))$
$(p \wedge \neg q) \otimes (q \otimes \neg p) \wedge (\neg q \wedge p) \otimes (\neg p \otimes q)$
$(p \wedge \neg q) \otimes (q \wedge \neg p) \wedge (\neg q \wedge p) \otimes (\neg p \wedge q)$
$(p \vee q) \wedge (p \vee \neg p) \wedge (\neg q \vee q) \wedge (\neg q \vee \neg p) \wedge (\neg q \vee \neg p) \wedge (\neg q \vee \neg p) \wedge (\neg q \vee$
$q) \wedge (p \vee \neg p) \wedge (p \vee q)$

Solution **??** [Example $(p \wedge r) \supset q$]

$$\begin{aligned}
CNF((p \wedge r) \supset q) \\
CNF((\neg(p \wedge r))\,) \otimes CNF(\,q) \\
(CNF((\neg p)\,) \otimes CNF(\,\neg r)) \otimes q \\
(\neg p \otimes \neg r) \otimes q \\
(\neg p \vee \neg r) \otimes q \\
(\neg p \vee q) \wedge (\neg r \vee q)
\end{aligned}$$

Check satisfiability cia CNN

Solution **??** Here is the solution by using DPLL:

1. (¬ p ∨ q) ∧ (¬ r ∨ q)

2. {{¬p, q}, {¬r, q} }

3. $\{\{\neg p, \top\}, \{\neg r, \top\}\}|_q$
4. $\{\}$

Solution **??** Here is the solution by using DPLL:

1. $(p \vee q) \wedge (p \vee \neg p) \wedge (\neg q \vee q) \wedge (\neg q \vee \neg p) \wedge (\neg q \vee \neg p) \wedge (\neg q \vee \neg p) \wedge (\neg q \vee q) \wedge (p \vee \neg p) \wedge (p \vee q)$
2. $\{\{p,q\},\{p,\neg p\},\{\neg q,q\},\{\neg q,\neg p\},\{\neg q,\neg p\},\{\neg q,\neg p\},\{\neg q,q\},\{p,\neg p\},\{p,q\}\}$
3. $\{\{\top,q\},\{\top,\bot\},\{\neg q,q\},\{\neg q,\bot\},\{\neg q,\bot\},\{\neg q,\bot\},\{\neg q,q\},\{\top,\bot\},\{\top,q\}\}|_p$
4. $\{\{\neg q,q\},\{\neg q\},\{\neg q\},\{\neg q\},\{\neg q,q\}\}$
5. $\{\{\top,\bot\},\{\top\},\{\top\},\{\top\},\{\top,\bot\}\}|_{\neg q}$
6. $\{\}$

Solution **??** Here is the solution by using DPLL:

1. $(q \vee \neg p) \wedge (q \vee \neg p)$
2. $\{\{q,\neg p\},\{q,\neg p\}\}$
3. $\{\{q,\top\},\{q,\top\}\}|_{\neg p}$
4. $\{\}$

Solution **??** Here is the solution by using DPLL:

1. $(q \vee \neg p) \wedge (\neg q \vee p) \wedge (p \vee q)$
2. $\{\{q,\neg p\},\{\neg q,p\},\{p,q\}\}$
3. $\{\{q,\top\},\{\neg q,\bot\},\{\bot,q\}\}|_{\neg p}$
4. $\{\{\neg q\},\{q\}\}$
5. $\{\{\bot\},\{\top\}\}|_q$
6. $\{\{\}\}$

Solution **??**

$\{\{p\},\{\neg p,\neg q\},\{\neg q,r\}\}$
$\{\{p\},\{\neg p,\neg q\},\{\neg q,r\}\}\ |_p$
$\{\{\top\}\ ,\{\ \bot,\neg q\},\{\neg q,r\}\}$
$\{\{\neg q\},\{\neg q,r\}\}$
$\{\{\neg q\},\{\neg q,r\}\}\ |_{\neg q}$
$\{\{\top\},\{\top,r\}\}$
$\{\}$

ϕ is satisfiable, and $I = \{p, \neg q\}$. The literal r is left undefined, because in order to satisfy the formula there is no need to evaluate it. This is an example of partial evaluation.

Solution **??**

$\{\{p\},\{\neg p\},\{\neg q,r\}\}$
$\{\{p\},\{\neg p\},\{\neg q,r\}\}|_p$
$\{\{\top\},\{\bot\},\{\neg q,r\}\}$
$\{\{\},\{\neg q,r\}\}$
$\{\ldots,\{\},\ldots\}$

The second clause cannot be verified, and thus the entire formula is not satisfiable.