$\mathcal{LOP}$ - Logic of Propositions

Basic Notions

Solution **1.1** ANSWER: ¬ ⊤ ∨ ≡ ⊃ ⊥ ∧ ⊨

Solution **1.2** ANSWER: 2,3.

Solution **1.3** Well formed formulas: 2. and 5.

Solution **1.4**

- An interpretation I(L) is such that for every formula A belonging to L we have A belonging to I(L) or ¬ A belonging to I(L). (**F**)
- A theory T is such that for every formula A belonging to L we have A belonging to T or ¬ A belonging to T. (**F**)
- A model M is an interpretation of a language L that satisfies all the formulas of T. (**T**)
- Every model M of a consistent theory T is such that T is a proper or improper subset of M. (**T**)
- An inconsistent theory has only one model. (**F**)
- Given a language L with N atomic propositions, a tautology (a valid formula) has 2^N models. (**T**)

Solution **1.5**

1. If T_1 and T_2 are PL theories, then, if $T_1 \models A$ and $T_2 \cup \{A\} \models B$ then $T_1 \cup T_2 \models B$. (**T**)
2. T is a theory in PL if and only if $T \models A$ implies that $A \in T$. (**T**)
3. A PL theory always contains an infinite set of formulas. (**T**)
4. If the PL formula A is satisfiable then $\neg A$ is unsatisfiable. (**F**)
5. If T_1 and T_2 are two PL theories then, $T_1 \models A$ and $T_1 \cup T_2 \models A$ if and only if T_2 is not empty. (**F**)

Informal to formal

Solution **1.6** We have the following:

1. $p \wedge q \rightarrow \neg r$
2. $p \rightarrow q$
3. $\neg(p \wedge \neg q)$ ..which is equivalent to $p \rightarrow q$

(NOTE) Notice how the precision of formal languages avoid the ambiguities of natural languages.

Solution **1.7** For both statements 1. and 2. we have $p \rightarrow q$.

Solution **1.8** We have the following.

1. $\neg A$
2. $A \wedge B$
3. $A \rightarrow \neg B$
4. $A \vee (\neg A \rightarrow B)$ logically equivalent to $A \vee B$
5. $(A \wedge B) \vee (\neg A \wedge \neg B)$ logically equivalent to $A \leftrightarrow B$

Solution **1.9** We have the following

1. $D \rightarrow B \wedge C$
2. $C \rightarrow \neg A \wedge \neg B$
3. $D \leftrightarrow (C \wedge \neg A)$
4. $D \rightarrow (\neg C \rightarrow A)$
5. $(\neg D \rightarrow C) \wedge (D \rightarrow \neg B)$
6. $A \rightarrow (\neg B \wedge \neg C \rightarrow D)$
7. $(A \wedge B \wedge C \leftrightarrow \neg D) \wedge (\neg A \wedge \neg B \rightarrow (D \leftrightarrow C))$

Solution **1.10** By now you should be able to do it without help.

Solution **1.11** By now you should be able to do it without help.

Solution **1.12**

1. $p \vee q \supset \neg r$
2. $p \supset q$
3. $\neg(p \wedge \neg q)$ which is equivalent to $p \supset q$!!!

Notice that the precision of formal language in your answers allows to avoid ambiguities of natural languages.

Solution **1.13** ANSWER:

- David ¬ Bruno
- Bruno
- David + Bruno
- Carlo ⊃ David
- (David ∧ ¬Bruno) ¬ (Bruno ∧ ¬David)
- ¬ Carlo ∧ ¬ David

Solution **1.14** ANSWER:

- David ⊃ (Bruno ∧ Carlo)
- (Bruno ∧ ¬Carlo) ⊃ Angelo
- (¬David ⊃ Carlo) ∧ (David ⊃ ¬Bruno)
- (¬Angelo ∧ ¬Bruno) ≡ Carlo
- (¬Bruno ∧ ¬Carlo) ⊃ Angelo
- (¬Bruno ∧ ¬Carlo) ≡ Angelo

Solution **1.15** Define the set of Propositions P, S, E such that:
P = "you play", S = "you study"; E = "you pass the exam"
The Theory T can be formalized as follows:

- (P ∧ S) ⊃ E

- $(P \wedge \neg S) \supset \neg E$
- $P \supset ((S \wedge E) \vee (\neg S \wedge \neg E))$

Solution **1.16** Define the set of Propositions RedBanana, YellowBanana, EatBanana, Crazy, Banana.
The Theory T can be formalized as follows:

- RedBanana ⊃ Banana
- YellowBanana ⊃ Banana
- EatBanana ⊃ YellowBanana
- ¬ EatBanana ⊃ Crazy

Solution **1.17** The LOP theory will include a proposition for each fact in the KG, and may contain some negations of propositions that are not expressed as facts in the KG. For instance:

- Theory#1
  - wasCreatedBy(TheMonaLisa, LeonardoDaVinci)
  - friendOf(Bob, Alice)
  - ¬ friendOf(Bob, Carol)
- Theory#2
  - TheMonaLisa_wasCreatedBy_LeonardoDaVinci
  - Bob_friendOf_Alice
  - ¬ Bob_friendOf_Carol

Solution **1.18** The sentence translates as P: Train + Plane. So, the correct answer is (4).

Solution **1.19** The phrase translates as P: Umbrella ∧ Wet. Consequently, the only correct answer is (1).

Solution **1.20** The sentence must necessarily be translated as an atomic proposition. Consequently, the only correct answer is (7).

Reasoning problems

Solution **1.21** v satisfies formulas 1., 3. and 4. v doesn't satisfy 2.

Solution **1.22** The argument is logically correct: if p means *I'm guilty* and q means *I must be punished*, then: $(p \rightarrow q) \wedge p \models q$ (by modus ponens).

Solution **1.23** The argument is not logically correct: $(p \rightarrow q) \wedge \neg p \not\models \neg q$. NOTE: consider for instance $v(p) = \mathrm{F}$ and $v(q) = \mathrm{T}$

Solution **1.24** By now you should be able to do it without help.

Solution **1.25** By now you should be able to do it without help.

Solution **1.26** By now you should be able to do it without help.

Solution **1.27** (Q) Compute the truth table of $(F \vee G) \wedge \neg(F \wedge G)$.

F	G	$F \vee G$	$F \wedge G$	$\neg(F \wedge G)$	$(F \vee G) \wedge \neg(F \wedge G)$
T	T	T	T	F	**F**
T	F	T	F	T	**T**
F	T	T	F	T	**T**
F	F	F	F	T	**F**

(A) The truth table is reported below. (NOTE) the formula models an exclusive or!

Solution **1.28** (Q) Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with φ) is satisfiable.

p	q	$p \rightarrow q$	$\neg q$	$p \rightarrow \neg q$	$(p \rightarrow q) \vee (p \rightarrow \neg q)$
T	T	T	F	F	**T**
T	F	F	T	T	**T**
F	T	T	F	T	**T**
F	F	T	T	T	**T**

(A) The formula is valid since it is satisfied by every interpretation. Being valid is also satisfiable

Solution **1.29** (Q) Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with φ) is satisfiable.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	φ
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	**T**
F	F	F	T	T	T	F	F

(A) There exists an interpretation satisfying φ, thus φ is satisfiable.

Solution **1.30** (Q) Use the truth tables method to determine whether the formula $\varphi : p \wedge \neg q \rightarrow p \wedge q$ is a logical consequence of the formula $\psi : \neg p$.

p	q	$\neg p$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \rightarrow p \wedge q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	**T**	F	F	**T**
F	F	**T**	F	F	**T**

(A) $\psi \models \varphi$ since each interpretation satisfying psi satisfies also φ.

Solution **1.31** (Q) Use the truth tables method to determine whether $p \rightarrow (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \wedge \neg q$	$p \rightarrow (q \wedge \neg q)$	$\neg p$
T	T	F	**F**	**F**
T	F	F	**F**	**F**
F	T	F	**T**	**T**
F	F	F	**T**	**T**

(A) The two formulas are equivalent since for every possible interpretation they evaluate to the same truth value.

Solution **1.32** By now you should be able to do it without help.

Solution **1.33** By now you should be able to do it without help.

Solution **1.34** By now you should be able to do it without help.

Solution **1.35** By now you should be able to do it without help.

Solution **1.36** By now you should be able to do it without help.

Solution **1.37**

- Compute all I
  - $\{P, Q, R, S\}$
  - $\{P, Q, R\}$
  - ...
- For each and any I, recursively replace each occurrence of each primitive proposition of the formula with the truth value assigned by I, and apply the definition for connectives.
- With $I = \{P\}$ we have the following

$$(True \wedge False) \vee (False \supset False)$$
$$False \vee True$$
$$True$$

Therefore the formula above is satisfiable.

Solution **1.38**

- Check if $(P \wedge Q) \vee (R \supset S)$ is valid.
- Compute all I
  - $\{P, Q, R, S\}$
  - $\{P, Q, R\}$
  - ...
- For each and any I, recursively replace each occurrence of each primitive proposition of the formula with the truth value assigned by I, and apply the definition for connectives.
- With $I = \{P, R\}$ we have the following

$$(True \wedge False) \vee (True \supset False)$$
$$False \vee False$$
$$False$$

Therefore the algorithm returns NO and we no longer have to check the other interpretations.

Solution **1.39**

p	q	r	$q \vee r$	$q \supset \neg p$	$\neg(r \wedge p)$	A	$\neg p$
T	T	T	T	F	F	F	F
T	T	F	T	F	T	F	F
T	F	T	T	T	F	F	F
T	F	F	F	T	T	F	F
F	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	F	T

Whenever A is true, $\neg p$ is also true, making it a logical consequence of A

Solution **1.40**

p	q	$q \wedge \neg q$	$p \supset (q \wedge \neg q)$	$\neg p$
T	T	F	F	F
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

The evaluated truth value is the same for every interpretation, therefore the formulas are logically equivalent.

Solution **1.41**

- ¬ A ∨ B is satisfiable
- (A ∧ B) ⊃ B is satisfiable
- (A ∧ B) ⊃ is valid
- If B is a tautology, then A ⊃ B is valid

Solution **1.42** To solve this exercise when can use a truth table

A	B	C	D	$(\neg(A \vee B) \wedge (C \wedge \neg D)) \supset (\neg C \vee B)$
T	T	T	T	T
T	T	T	F	T
T	T	F	T	T
T	T	F	F	T
T	F	T	T	T
T	F	T	F	T
T	F	F	T	T
T	F	F	F	T
F	T	T	T	T
F	T	T	F	T
F	T	F	T	T
F	T	F	F	T
F	F	T	T	T
F	F	T	F	F
F	F	F	T	T
F	F	F	F	T

We can quickly fill the truth values of the formula for the rows where A or B are true. In fact, if you look at the first member of the formula $(\neg(A \vee B)$, this is true only if both A and B are false. Since this member is in conjunction ($\wedge$) with the other, the premise of the implication will always be false if either A or B are true. If the premise of the implication is false, we have that the entire implication evaluates to true. Then we have the last four cases that are to be evaluated manually.

Solution **1.43** ANSWER:

	VARIABLES		(1)	(2)	(3)
	A	B	A∧B	A∨B	A ≡ B
POSSIBLE ASSIGNMENTS	T	T	T	T	T
	T	F	F	T	F
	F	T	F	T	F
	F	F	F	F	T

Solution **1.44** ANSWER:
See here for (1).
Check yourself for (2), (3), (4).

	A	B	C	A ∧ ¬ B
	T	T	T	F
MODEL →	T	F	T	T
	F	T	T	F
	F	F	T	F
	T	T	F	F
MODEL →	T	F	F	T
	F	T	F	F
	F	F	F	F

Solution **1.45** ANSWER:
You can rewrite P as ¬ (A ∨ B) ∨ ¬ C that is equivalent to (¬ A ∧ ¬ B) ∨ ¬ C

MODELS

A	B	C	P
T	T	T	F
T	F	T	F
F	T	T	F
F	F	T	T
T	T	F	T
T	F	F	T
F	T	F	T
F	F	F	T

Solution **1.46** ANSWER:

A	B	P
T	T	F
T	F	F
F	T	F
F	F	F

Solution **1.47** ANSWER:
Notice that you can rewrite it as ¬¬(A ⊃ B) ∨ (A ∧ ¬B) that is equivalent to(¬A ∨ B) ∨ (A ∧ ¬B)

A	B	P
T	T	T
T	F	T
F	T	T
F	F	T

Solution **1.48** PROOF:
Let use define two propositions Z = "Zoey is a knight" and M = "Mel is a knight". The two sentences above can be translated in LOP as follows:

- Z ⊃ ¬M
- M ⊃ Z ∧ M

We can use truth tables to prove that there are only two possible answers:

- Both lie, i.e. they are both knaves, i.e. I(Z) = F and I(M) = F
- Zoey tells the true (is a Knight) and Mel lies (is a knave), i.e. I(Z) = T and I(M) = F

Solution **1.49** PROOF:
((A ∨¬E) ∧ (¬B ∨¬D)) ⊃ B
≡ (A ∨¬E) ⊃ ((¬B∨¬D) ⊃ B) (by Implication and conjunction)
≡(A ∨¬E) ⊃ (¬(¬B∨¬D) ∨ B) (by Implication and disjunction)
≡(A ∨¬E) ⊃ ((B∧D) ∨ B) (by De Morgan laws)
≡ ¬ (A ∨¬E) ∨ ((B∧D) ∨ B) (by Implication and disjunction)
≡(¬ A ∧E) ∨ ((B∧D) ∨ B) (by De Morgan laws)

Solution **1.50** PROOF:
If A ⊨ B ∧ C, for all I such that I(A) = True it should be I(B ∧ C) = True.

However, by definition this means that I(B) = True and I(C) = True.
Therefore both A $\models$ B and A $\models$ C.

Solution **1.51** PROOF:
RECALL: I = X, Y means that I(X) = T, I(Y) = T, I(Z) = F, while I' = X means that I(X) = T, I(Y) = F, I(Z) = F.
$I(A) = I(X \wedge \neg Z) = I(X) \wedge I(\neg Z) = T$
$I'(A) = I'(X \wedge \neg Z) = I'(X) \wedge I'(\neg Z) = T$

Solution **1.52** PROOF:
A way to prove validity is to show that P $\models \top$.
This can be done by applying well known tautologies (e.g. De Morgan).
$(A \supset B) \supset (\neg A \vee B)$
$\equiv \neg(A \supset B) \vee (\neg A \vee B)$ (by Implication and disjunction)
$\equiv \neg(\neg A \vee B) \vee (\neg A \vee B)$ (by Implication and disjunction)
$\equiv \top$ (by The law of the excluded middle)

Solution **1.53** PROOF:
Let $X = A \vee B$, $Y = \neg D \vee E$, then we can rewrite:
$P = X \wedge (\neg C \vee Y)$; $Q1 = X$; $Q2 = (X \vee C) \wedge (\neg B \vee \neg C \vee Y)$; $Q3 = X \wedge Y$

P $\models$ Q1 is obvious.
Since $X \models X \vee C$ and $(\neg C \vee Y) \models (\neg B \vee \neg C \vee Y)$, then P $\models$ Q2.
Since $Y \models (\neg C \vee Y)$, then Q3 $\models$ P (and not the vice versa).

Solution **1.54** Answer to (1): First compute the truth tables for all the propositions above. Then, list all rows for which both P and Qi are true.
Answer to (2): Check whether there is any assignment for which all the sentences above are true.

Solution **1.55** ANSWER: b, e, f.

Solution **1.56** ANSWER: The models of T1 are M1, M4. The models of T2 are M3.
Note that M1 = A, B, M4 = and M3 = B.
Therefore the true sentences are: a, b, f, h.

Solution **1.57** ANSWER: a, e.

Solution **1.58** ANSWER: b, d, f.

Solution **1.59**

1. True, as per the definition of $\mathcal{LOP}$ interpretation function
2. True, as per the definition of $\mathcal{LOP}$ interpretation function
3. False, as per the definition of interpretation function of $\mathcal{LOP}$, the Translate translation function is one-to-one.
4. False, in $\mathcal{LODE}$ for each theory there is one and only one model that encodes its intended meaning.

Solution **1.60** By now you should be able to do it yourself.

Solution **1.61** By now you should be able to do it yourself.